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On the variety generated by all semigroups of order three ☆

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ABSTRACT

Denote by $\mathbf{S}_n$ the variety generated by all semigroups of order n. Marcel Jackson proved that the variety $\mathbf{S}_n$ contains uncountably many subvarieties if $n \geqslant 4$, and it follows from existing results that the variety $\mathbf{S}_2$ contains precisely 32 subvarieties. However, the number of subvarieties of the variety $\mathbf{S}_3$ has been unknown. The main aim of the present article is to address this problem. It is shown that all subvarieties of the variety $\mathbf{S}_3$ are finitely based. Consequently, the variety $\mathbf{S}_3$ contains countably infinitely many subvarieties.

1. Introduction

Semigroups of small order play crucial roles in the study of semigroup varieties. The presence or absence of certain small semigroups in a variety often controls important structural and equational properties satisfied by the variety. For instance, any locally finitely variety containing Perkins' semigroup B_2^1 of order six is non-finitely based [16], while any variety that does not contain the null semigroup of order two consists entirely of completely regular semigroups. Refer to [18, Table 1] for a list of other properties satisfied by a variety that excludes a combination of other semigroups of order two.

Over the years, small semigroups have been studied in several ways. Most notably, all semigroups of order five [20,21] or less [3,13] had been shown to be finitely based; this result confirmed that

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the semigroup B_2^1 is minimal with respect to being non-finitely based. On the other hand, a semigroup with as few as four elements can generate a variety that is *finitely universal* [8] in the sense that its subvariety lattice contains an isomorphic copy of any finite lattice. (In contrast, the variety generated by any finite group [12] or finite associative ring [7,9] is finitely based and contains only finitely many subvarieties.) Refer to the surveys [17–19] for more information on semigroup varieties and the lattices they constitute.

Small semigroups have also been studied collectively. For each integer $n \geqslant 2$, let $\mathbf{S}_n$ be the variety generated by all semigroups of order n. The varieties $\mathbf{S}_2$ and $\mathbf{S}_3$ are finitely based (see [19, Section 10]), and Volkov [25] proved that the variety $\mathbf{S}_n$ is non-finitely based for all $n \geqslant 5$. But it is still unknown if the variety $\mathbf{S}_4$ is finitely based. As for the number of subvarieties in $\mathbf{S}_n$, Jackson [5] proved that the variety $\mathbf{S}_n$ contains uncountably many subvarieties if $n \geqslant 4$. He also demonstrated that the variety $\mathbf{S}_2$ contains only countably many subvarieties. These results naturally led him to question whether or not the variety $\mathbf{S}_3$ contains uncountably many subvarieties [5, Question 3.18].

A result of Vernikov and Volkov [22] actually implies that the variety $\mathbf{S}_2$ contains precisely 32 subvarieties (see Proposition 3.4). The situation with the variety $\mathbf{S}_3$, however, is not as simple, since there exists a monoid of order three that generates a variety with infinitely many subvarieties (see Section 3.2). Nevertheless, a negative answer to the aforementioned question of Jackson follows from the main result of the present article.

Theorem 1.1. *Every subvariety of* $\mathbf{S}_3$ *is finitely based.*

Since only countably many finite sets of identities exist up to letter substitution, a variety that contains only finitely based subvarieties can contain at most countably many subvarieties. Consequently, the variety $\mathbf{S}_3$ contains countably infinitely many subvarieties. It also follows that the variety $\mathbf{S}_3$ is finitely universal (it is known that if $n \geqslant 4$, then the variety $\mathbf{S}_n$ is finitely universal [8]). Table 1 summarizes all the aforementioned results regarding the variety $\mathbf{S}_n$.

Table 1
Some properties of the variety $\mathbf{S}_n$.

	$n = 2$	$n = 3$	$n = 4$	$n \geqslant 5$
$\mathbf{S}_n$ is finitely based	Yes	Yes	Unknown	No
Number of subvarieties of $\mathbf{S}_n$	$32^{(\dagger)}$	$\aleph_0$	$2^{\aleph_0}$	$2^{\aleph_0}$
$\mathbf{S}_n$ is finitely universal	No	Yes$^{(\ddagger)}$	Yes	Yes

There are nine sections in this article. Section 2 contains notation and background material. Section 3 contains information on semigroups of order up to three and specifically addresses the results (†) and (‡). Section 4 establishes a finite basis for the variety $\mathbf{S}_3$. Section 5 presents the main arguments in the proof of Theorem 1.1. Technically involved results that are required for this proof are deferred to the last four sections.

2. Preliminaries

Most of the notation and background material of this article are given in this section. Refer to the monograph [1] for any undefined terminology.

Let $\mathcal{X}$ be a fixed countably infinite alphabet throughout. For any subset $\mathcal{A}$ of $\mathcal{X}$, denote by $\mathcal{A}^*$ the free monoid over $\mathcal{A}$. Elements of $\mathcal{X}$ and $\mathcal{X}^*$ are referred to as *letters* and *words* respectively.

Let x be any letter and $\mathbf{w}$ be any word. Then

- the *content* of $\mathbf{w}$, denoted by $\mathsf{C}(\mathbf{w})$, is the set of letters occurring in $\mathbf{w}$;
- the *head* of $\mathbf{w}$, denoted by $\mathsf{h}(\mathbf{w})$, is the first letter occurring in $\mathbf{w}$;
- the *tail* of $\mathbf{w}$, denoted by $\mathsf{t}(\mathbf{w})$, is the last letter occurring in $\mathbf{w}$;
- the *initial part* of $\mathbf{w}$, denoted by $\mathsf{ip}(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the first occurrence of each letter;

- the *final part* of $\mathbf{w}$, denoted by $\mathsf{fp}(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the last occurrence of each letter;
- the *multiplicity* of x in $\mathbf{w}$, denoted by $\mathsf{m}(x,\mathbf{w})$, is the number of times x occurs in $\mathbf{w}$;
- x is *simple* in $\mathbf{w}$ if $\mathsf{m}(x,\mathbf{w})=1$.

A word is said to be *simple* if all of its letters are simple in it.

An identity is typically written as $\mathbf{u}\approx\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are nonempty words. Let Π be any set of identities. The deducibility of an identity $\mathbf{u}\approx\mathbf{v}$ from Π is indicated by $\Pi\vdash\mathbf{u}\approx\mathbf{v}$ or $\mathbf{u}\overset{\Pi}{\approx}\mathbf{v}$. The variety *defined* by Π is the class of all semigroups that satisfy all identities in Π; in this case, Π is said to be a *basis* for the variety. The subvariety of $\mathbf{S}_3$ defined by Π is denoted by $\mathbf{S}_3\{\Pi\}$. A variety is *finitely based* if it possesses a finite basis.

Let $\mathbf{u}\approx\mathbf{v}$ be any identity and x be any letter. Then

- $\mathbf{u}\approx\mathbf{v}$ is *n-balanced at x* if $\mathsf{m}(x,\mathbf{u})\equiv\mathsf{m}(x,\mathbf{v}) \pmod n$;
- $\mathbf{u}\approx\mathbf{v}$ is *n-balanced* if it is n-balanced at every letter;
- $\mathbf{u}\approx\mathbf{v}$ is *balanced at x* if $\mathsf{m}(x,\mathbf{u})=\mathsf{m}(x,\mathbf{v})$;
- $\mathbf{u}\approx\mathbf{v}$ is *balanced* if it is balanced at every letter.

A *permutation identity* is a balanced identity $\mathbf{u}\approx\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are distinct simple words.

Lemma 2.1. *(See Perkins [13].) Any periodic variety that satisfies some permutation identity is finitely based.*

The number $|\mathsf{C}(\mathbf{w})|$ of distinct letters in a word $\mathbf{w}$ obviously cannot exceed the length $|\mathbf{w}|$ of $\mathbf{w}$. Define the difference $|\mathbf{w}|-|\mathsf{C}(\mathbf{w})|$ to be the *level* of the word $\mathbf{w}$. The *level* of an identity $\mathbf{u}\approx\mathbf{v}$ is the maximum of the level of $\mathbf{u}$ and $\mathbf{v}$. Denote by Λ_n the set of all identities over $\mathcal{X}$ of level at most n.

Lemma 2.2. *(See Volkov [24].) Let n be any fixed integer and Λ be any subset of Λ_n. Then the variety defined by Λ is finitely based.*

3. Varieties generated by small semigroups

This section contains information on semigroups of order two and three. Varieties generated by these semigroups and their bases are long known results and will be given here.

Let L_2 be the left-zero semigroup of order two, R_2 be the right-zero semigroup of order two, and Y_2 be the semilattice of order two:

$$L_2=\langle a,b \mid a^2=ab=a,\ b^2=ba=b\rangle,$$
$$R_2=\langle a,b \mid a^2=ba=a,\ b^2=ab=b\rangle,$$
$$Y_2=\{0,1\}.$$

Let N_2 and N_3 be the aperiodic monogenic semigroups of order two and three respectively:

$$N_2=\langle a \mid a^2=a^3\rangle,$$
$$N_3=\langle a \mid a^3=a^4\rangle.$$

For any $n\geqslant 2$, let $\mathbb{Z}_n$ be the cyclic group of order n. The semigroup B_2^1 introduced in Section 1 is the monoid obtained by adjoining a neutral element to the Brandt semigroup

$$B_2=\langle a,b \mid a^2=b^2=0,\ aba=a,\ bab=b\rangle$$

of order five. It is routine to verify that $B_0 = \{0, b, ab, ba\}$, $C = \{0, ab, b\}$, and $D = \{0, ab, a\}$ are subsemigroups of B_2. The semigroups C and D are easily shown to be anti-isomorphic. If S is any semigroup that is not a monoid, then S^1 denotes the monoid obtained from S by adjoining a neutral element.

The variety generated by a semigroup S is denoted by $\operatorname{var} S$. A minimal nontrivial variety is said to be an *atom*.

Lemma 3.1. *(See Kalicki and Scott [6].) The varieties* $\operatorname{var} L_2$, $\operatorname{var} R_2$, $\operatorname{var} N_2$, $\operatorname{var} Y_2$, *and* $\operatorname{var} \mathbb{Z}_p$ *for all prime integers p constitute the set of all atoms in the lattice of semigroup varieties.*

For any $\mathsf{x} \in \{\mathsf{C}, \mathsf{ip}, \mathsf{fp}\}$, an identity $\mathbf{u} \approx \mathbf{v}$ is said to be x-*compliant* if $\mathsf{x}(\mathbf{u}) = \mathsf{x}(\mathbf{v})$.

Lemma 3.2.

(i) *An identity holds in the monoid* L_2^1 *if and only if it is* ip*-compliant.*
(ii) *An identity holds in the monoid* R_2^1 *if and only if it is* fp*-compliant.*
(iii) *An identity holds in the semilattice* Y_2 *if and only if it is* C*-compliant.*
(iv) *An identity holds in the group* $\mathbb{Z}_n$ *if and only if it is n-balanced.*
(v) *An identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the semigroup* N_3 *if and only if either* $\mathbf{u} \approx \mathbf{v}$ *is balanced or* $|\mathbf{u}|, |\mathbf{v}| \geqslant 3$.
(vi) *An identity* $\mathbf{u} \approx \mathbf{v}$ *holds in the monoid* N_2^1 *if and only if for each* $x \in \mathcal{X}$, *either* $\mathsf{m}(x, \mathbf{u}) = \mathsf{m}(x, \mathbf{v}) = 1$ *or* $\mathsf{m}(x, \mathbf{u}), \mathsf{m}(x, \mathbf{v}) \geqslant 2$.

Proof. Parts (i) to (iv) are well known (see, for example, [15]). Parts (v) and (vi) are easy to verify. □

3.1. Semigroups of order two

Lemma 3.3. *(See Vernikov and Volkov [22, Proposition 1].) Let* $\mathbf{V}$ *be any variety that is the join of n distinct atoms. Then the lattice of subvarieties of* $\mathbf{V}$ *is isomorphic to a Boolean algebra of order* 2^n.

Proposition 3.4. *The variety* $\mathbf{S}_2$ *contains precisely* 32 *subvarieties.*

Proof. The semigroups L_2, R_2, N_2, Y_2, and $\mathbb{Z}_2$ are, up to isomorphism, the only semigroups of order two (see, for example, [14]) so that by Lemma 3.1, the variety $\mathbf{S}_2$ is the join of five distinct atoms. The result now follows from Lemma 3.3. □

3.2. Semigroups of order three

Up to isomorphism and anti-isomorphism, there are 18 semigroups of order three (see, for example, [14]). Multiplication tables of these semigroups are listed in the first column of Table 2. Bases for these semigroups and the varieties these semigroups generate can be found in [3,4] and are also summarized in Table 2. It is routine to deduce that the variety $\mathbf{S}_3$ is generated by the semigroups C, D, L_2^1, R_2^1, N_2^1, N_3, $\mathbb{Z}_2$, and $\mathbb{Z}_3$.

Let $\mathbf{H}$ be the variety defined by the identity $xyx \approx z^2$. By inspecting the basis for the variety $\mathbf{S}_3$ found in Theorem 4.1, it is easy to show that the variety $\mathbf{H}$ is contained in $\mathbf{S}_3$. Since the variety $\mathbf{H}$ is finitely universal [23], it follows from Theorem 1.1 that the variety $\mathbf{S}_3$ is finitely universal and contains countably infinitely many subvarieties.

A variety is said to be *small* if it contains finitely many subvarieties. By Lemmas 3.1 and 3.3, the varieties in Table 2 that are either atoms or joins of atoms are small. The variety $\operatorname{var} L_2^1$ is well known to be small (see [15]), while the varieties $\operatorname{var} N_3$ and $\operatorname{var} C$ are small by a result of Malyshev [10]. However, the remaining variety $\operatorname{var} N_2^1$ is non-small [2] (see also [4, Fig. 5(b)]). Therefore it seems that the presence of the semigroup N_2^1 in the variety $\mathbf{S}_3$ is the reason for the complex structure of the subvariety lattice of $\mathbf{S}_3$. It is thus reasonable to conjecture that a subvariety of $\mathbf{S}_3$ is small if and only if

Table 2
Semigroups of order three. The underlying set of each semigroup is $\{1, 2, 3\}$, and each table is given by a 3×3 matrix where the (i, j)-entry denotes the product of the elements i and j.

Semigroup	Basis	Variety generated
111 111 111	$xy \approx zt$	$\operatorname{var} N_2$
111 111 112	$xyz \approx hkt$, $xy \approx yx$	$\operatorname{var} N_3$
111 121 111	$x^2y \approx xy$, $xy \approx yx$	$\operatorname{var} N_2 \vee \operatorname{var} Y_2$
111 121 113	$x^2 \approx x$, $xy \approx yx$	$\operatorname{var} Y_2$
111 121 131	$xy^2 \approx xy$, $x^2y^2 \approx y^2x^2$	$\operatorname{var} C$
111 121 333	$x^2 \approx x$, $xyz \approx xzy$	$\operatorname{var} L_2 \vee \operatorname{var} Y_2$
111 122 122	$x^2y \approx xy$, $xy \approx yx$	$\operatorname{var} N_2 \vee \operatorname{var} Y_2$
111 122 123	$x^2 \approx x$, $xy \approx yx$	$\operatorname{var} Y_2$
111 122 133	$x^2 \approx x$, $xyz \approx xzy$	$\operatorname{var} L_2 \vee \operatorname{var} Y_2$
111 123 131	$x^3 \approx x^2$, $xy \approx yx$	$\operatorname{var} N_2^1$
111 123 132	$x^3 \approx x$, $xy \approx yx$	$\operatorname{var} Y_2 \vee \operatorname{var} \mathbb{Z}_2$
111 123 333	$x^2 \approx x$, $xyx \approx xy$	$\operatorname{var} L_2^1$
111 222 111	$xyz \approx xy$	$\operatorname{var} L_2 \vee \operatorname{var} N_2$
111 222 333	$xy \approx x$	$\operatorname{var} L_2$
113 113 331	$x^2yz \approx yz$, $xy \approx yx$	$\operatorname{var} N_2 \vee \operatorname{var} \mathbb{Z}_2$
113 123 331	$x^3 \approx x$, $xy \approx yx$	$\operatorname{var} Y_2 \vee \operatorname{var} \mathbb{Z}_2$
122 211 211	$x^2yz \approx yz$, $xy \approx yx$	$\operatorname{var} N_2 \vee \operatorname{var} \mathbb{Z}_2$
123 231 312	$x^3y \approx y$, $xy \approx yx$	$\operatorname{var} \mathbb{Z}_3$

it does not contain the semigroup N_2^1. However, this conjecture is false. Consider the semigroup B_0; it generates a finitely universal variety [8]. By inspecting the basis for $\mathbf{S}_3$ found in Theorem 4.1, one can easily check that each identity in the basis holds in B_0, hence $\operatorname{var} B_0$ is a subvariety of $\mathbf{S}_3$. The identity $xy^2x \approx xyx$ holds in the semigroup B_0 but fails in the semigroup N_2^1. Therefore the variety $\operatorname{var} B_0$ is not only a non-small but finitely universal subvariety of $\mathbf{S}_3$ that does not contain the semigroup N_2^1.

4. A finite basis for $\mathbf{S}_3$

The present section establishes a finite basis for the variety $\mathbf{S}_3$. (The finite basis property of the variety $\mathbf{S}_3$ was first announced in [19, Section 10] as a result of Dvoskina. However, neither a proof of this result nor a finite basis for $\mathbf{S}_3$ was given.)

Theorem 4.1. *The variety* $\mathbf{S}_3$ *is defined by the identities*

$$x^8y \approx x^2y, \qquad xy^8 \approx xy^2, \qquad x^7yx \approx xyx^7 \approx xyx, \qquad xyx^6zx \approx xyzx, \tag{4.1}$$

$$x^2yx \approx xyx^2, \qquad xyxzx \approx x^2yzx, \tag{4.2}$$

$$xhyxty \approx xhxyty, \qquad xhyxy \approx xhxy^2, \qquad xyxty \approx x^2yty, \qquad xyxy \approx x^2y^2. \tag{4.3}$$

The proof of Theorem 4.1 is given at the end of the section.

Most of the equational deductions in this article are deductions within the equational theory of $\mathbf{S}_3$. Therefore, it will be convenient to refer to the identities in Theorem 4.1 collectively by Ⓢ, that is,

$$Ⓢ = \{(4.1), (4.2), (4.3)\}.$$

For any sets Π_1 and Π_2 of identities, the deduction $Ⓢ \cup \Pi_1 \vdash \Pi_2$ is abbreviated to $\Pi_1 \Vdash \Pi_2$.

Any word $\mathbf{w}$ can always be uniquely written in the form

$$\mathbf{w} = x_1^{e_1}x_2^{e_2}\cdots x_k^{e_k}$$

where $e_i \geqslant 1$ and $x_i \neq x_{i+1}$ for all i. The sequence $x_1x_2\cdots x_k$ is said to be the *appearance sequence* of $\mathbf{w}$. The number of times a letter x appears in the sequence $x_1x_2\cdots x_k$ is denoted by $\mathsf{a}(x, \mathbf{w})$.

Example 4.2. Suppose that $\mathbf{w} = x^2zxy^2xtzx^3t^4$. Then

(a) the appearance sequence of $\mathbf{w}$ is $xzxyxtzxt$;
(b) $\mathsf{a}(x, \mathbf{w}) = 4$, $\mathsf{a}(y, \mathbf{w}) = 1$, and $\mathsf{a}(z, \mathbf{w}) = 2 = \mathsf{a}(t, \mathbf{w})$;
(c) $\mathsf{m}(x, \mathbf{w}) = 7$, $\mathsf{m}(y, \mathbf{w}) = \mathsf{m}(z, \mathbf{w}) = 2$, and $\mathsf{m}(t, \mathbf{w}) = 5$.

A word $\mathbf{w}$ is said to be in *canonical form* if all of the following conditions hold:

(CF1) $\mathsf{a}(x, \mathbf{w}) \leqslant 2$ for any $x \in \mathcal{X}$,
(CF2) if $|\mathsf{C}(\mathbf{w})| = 1$, then $\mathbf{w} \in \{x, x^2, \ldots, x^8\}$ for some $x \in \mathcal{X}$,
(CF3) if $|\mathsf{C}(\mathbf{w})| \geqslant 2$, then $\mathsf{m}(x, \mathbf{w}) \in \{1, 2, \ldots, 7\}$ for any $x \in \mathsf{C}(\mathbf{w})$,
(CF4) if $\mathbf{w} = \mathbf{w}_1x^e\mathbf{w}_2x^f\mathbf{w}_3$ for some $x \in \mathcal{X}$, $e, f \geqslant 1$, and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in (\mathcal{X}\backslash\{x\})^*$ with $\mathbf{w}_2 \neq \emptyset$, then $f = 1$, $\mathsf{h}(\mathbf{w}_2) \notin \mathsf{C}(\mathbf{w}_1)$, and $\mathsf{t}(\mathbf{w}_2) \notin \mathsf{C}(\mathbf{w}_3)$.

An identity $\mathbf{u} \approx \mathbf{v}$ is *canonical* if the words $\mathbf{u}$ and $\mathbf{v}$ are in canonical form.

Lemma 4.3. *Let* $\mathbf{w}$ *be any word. Then the deduction* $Ⓢ \vdash \mathbf{w} \approx \mathbf{w}'$ *holds for some word* $\mathbf{w}'$ *in canonical form.*

Proof. It suffices to apply the identities Ⓢ to the word $\mathbf{w}$ until it satisfies all of the conditions (CF1)–(CF4). First, condition (CF1) is satisfied by applying the identities (4.2). Then conditions (CF2) and (CF3) are satisfied by applying the identities (4.1). Finally, condition (CF4) is satisfied by applying the identities (4.2) and (4.3). □

Lemma 4.4. *Let* $\mathbf{u} \approx \mathbf{v}$ *be any canonical identity that is* ip–fp-*compliant. Then*

(a) $\mathsf{a}(x, \mathbf{u}) = \mathsf{a}(x, \mathbf{v})$ *for all* $x \in \mathcal{X}$;
(b) *the appearance sequences of* $\mathbf{u}$ *and* $\mathbf{v}$ *coincide.*

Proof. Since the identity $\mathbf{u} \approx \mathbf{v}$ is ip–fp-compliant, $\mathsf{a}(x, \mathbf{u}) = 0$ if and only if $\mathsf{a}(x, \mathbf{v}) = 0$. Suppose that $\mathsf{a}(x, \mathbf{u}) = 2$ and $\mathsf{a}(x, \mathbf{v}) = 1$ for some $x \in \mathcal{X}$, say

$$\mathbf{u} = \mathbf{u}_1 x^e \mathbf{u}_2 x \mathbf{u}_3 \quad \text{and} \quad \mathbf{v} = \mathbf{v}_1 x^f \mathbf{v}_2$$

for some $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2 \in (\mathcal{X}\backslash\{x\})^*$ and $e, f \geqslant 1$ such that $\mathbf{u}_2 \neq \emptyset$. Let

$$\mathsf{ip}(\mathbf{u}_2) = y_1 \cdots y_k.$$

Then $y_1 \notin \mathsf{C}(\mathbf{u}_1)$ by condition (CF4) so that $\mathsf{ip}(\mathbf{u}) = \cdots xy_1 \cdots$. Since $\mathsf{ip}(\mathbf{v}) = \mathsf{ip}(\mathbf{u})$, the letter y_1 occurs in $\mathbf{v}_2$. Hence $\mathsf{fp}(\mathbf{v}) = \cdots x \cdots y_1 \cdots$. Since $\mathsf{fp}(\mathbf{u}) = \mathsf{fp}(\mathbf{v})$, the letter y_1 must occur in $\mathbf{u}_3$. A similar argument can be repeated to show that the letters $y_2, \ldots, y_k$ also occur in $\mathbf{u}_3$. This implies that $\mathsf{t}(\mathbf{u}_2) = y_k \in \mathsf{C}(\mathbf{u}_3)$, contradicting condition (CF4). A similar contradiction follows if $\mathsf{a}(x, \mathbf{u}) = 1$ and $\mathsf{a}(x, \mathbf{v}) = 2$ for some $x \in \mathcal{X}$. Therefore (a) must hold.

Suppose that the appearance sequences of $\mathbf{u}$ and $\mathbf{v}$ are different because of the order of appearance of the letters y and z. If either $\mathsf{a}(y, \mathbf{u}) = \mathsf{a}(y, \mathbf{v}) = 1$ or $\mathsf{a}(z, \mathbf{u}) = \mathsf{a}(z, \mathbf{v}) = 1$, then $\mathbf{u} \approx \mathbf{v}$ is either non-ip-compliant or non-fp-compliant, contradicting the assumption. Therefore

$$\mathsf{a}(y, \mathbf{u}) = \mathsf{a}(y, \mathbf{v}) = \mathsf{a}(z, \mathbf{u}) = \mathsf{a}(z, \mathbf{v}) = 2. \tag{4.4}$$

Assume without loss of generality that

$$\mathbf{u} = \mathbf{u}_1 y^{e_1} \mathbf{u}_2 y \mathbf{u}_3 z^{f_1} \mathbf{u}_4 z \mathbf{u}_5 \quad \text{and} \quad \mathbf{v} = \mathbf{v}_1 y^{e_2} \mathbf{v}_2 z^{f_2} \mathbf{v}_3 y \mathbf{v}_4 z \mathbf{v}_5$$

for some $\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_5, \mathbf{v}_5 \in (\mathcal{X}\backslash\{y, z\})^*$ and $e_1, f_1, e_2, f_2 \geqslant 1$. Note that both $\mathbf{u}_2$ and $\mathbf{u}_4$ are nonempty words due to (4.4). Further, the word $\mathbf{v}_3$ is also nonempty by condition (CF4), say $\mathsf{ip}(\mathbf{v}_3) = x_1 \cdots x_r$. First suppose that $\mathsf{a}(x_1, \mathbf{u}) = \mathsf{a}(x_1, \mathbf{v}) = 1$. Then $\mathsf{ip}(\mathbf{u}) = \mathsf{ip}(\mathbf{v}) = \cdots y \cdots z x_1 \cdots$ so that $x_1 \in \mathsf{C}(\mathbf{u}_4\mathbf{u}_5)$. It follows that

$$\mathsf{fp}(\mathbf{u}) = \cdots y \cdots x_1 \cdots \neq \cdots x_1 \cdots y \cdots = \mathsf{fp}(\mathbf{v}),$$

contradicting the fp-compliance of $\mathbf{u} \approx \mathbf{v}$. Hence $\mathsf{a}(x_1, \mathbf{u}) = \mathsf{a}(x_1, \mathbf{v}) = 2$ necessarily. In this case, it follows from condition (CF4) that $x_1 \notin \mathsf{C}(\mathbf{v}_1\mathbf{v}_2)$. Then $\mathsf{ip}(\mathbf{u}) = \mathsf{ip}(\mathbf{v}) = \cdots y \cdots z x_1 \cdots$ so that $x_1 \in \mathsf{C}(\mathbf{u}_4\mathbf{u}_5)$. Now $\mathsf{fp}(\mathbf{v}) = \mathsf{fp}(\mathbf{u}) = \cdots y \cdots x_1 \cdots$ implies that $x_1 \in \mathsf{C}(\mathbf{v}_4\mathbf{v}_5)$. A similar argument can be repeated to deduce that $x_2, \ldots, x_r \in \mathsf{C}(\mathbf{v}_4\mathbf{v}_5)$. It follows that $\mathsf{t}(\mathbf{v}_3) = x_r \in \mathsf{C}(\mathbf{v}_4\mathbf{v}_5)$, contradicting condition (CF4). Consequently, the letters y and z do not exist and (b) must hold. $\square$

Lemma 4.5. *The following conditions on a canonical identity* $\mathbf{u} \approx \mathbf{v}$ *are equivalent*:

(a) $\mathbf{u} \approx \mathbf{v}$ *holds in the variety* $\mathbf{S}_3$;
(b) $\mathbf{u} \approx \mathbf{v}$ *is balanced and* ip–fp-*compliant*;
(c) $\mathbf{u} \approx \mathbf{v}$ *is trivial.*

Proof. (a) implies (b). The variety $\mathbf{S}_3$ contains the semigroups L_2^1 and R_2^1 so that the identity $\mathbf{u} \approx \mathbf{v}$ is ip–fp-compliant by Lemma 3.2, parts (i) and (ii). Therefore it remains to show that the identity $\mathbf{u} \approx \mathbf{v}$ is balanced. Since the variety $\mathbf{S}_3$ contains the semigroups N_3, N_2^1, and $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, the identity $\mathbf{u} \approx \mathbf{v}$ satisfies all the properties in parts (iv)–(vi) of Lemma 3.2 with $n = 6$. It follows that the identity $\mathbf{u} \approx \mathbf{v}$ is C-compliant and 6-balanced. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ is non-balanced at some $x \in \mathcal{X}$. Then $\mathsf{m}(x, \mathbf{u}), \mathsf{m}(x, \mathbf{v}) \geqslant 2$ by Lemma 3.2(vi). Since the 6-balanced identity $\mathbf{u} \approx \mathbf{v}$ is non-balanced at x, it follows that $\{\mathsf{m}(x, \mathbf{u}), \mathsf{m}(x, \mathbf{v})\} = \{2, 8\}$. Hence by condition (CF3), it is impossible for $|\mathsf{C}(\mathbf{u})| = |\mathsf{C}(\mathbf{v})| \geqslant 2$. If $|\mathsf{C}(\mathbf{u})| = |\mathsf{C}(\mathbf{v})| = 1$, then $\{\mathbf{u}, \mathbf{v}\} = \{x^2, x^8\}$, but this contradicts Lemma 3.2(v) since the identity $x^2 \approx x^8$ is not balanced and $|x^2| \ngeqslant 3$. Consequently, the identity $\mathbf{u} \approx \mathbf{v}$ must be balanced.

(b) implies (c). It follows from Lemma 4.4 that $\mathsf{a}(x, \mathbf{u}) = \mathsf{a}(x, \mathbf{u})$ for all $x \in \mathcal{X}$ and the appearance sequences of $\mathbf{u}$ and $\mathbf{v}$ coincide. It is then easy to see the identity $\mathbf{u} \approx \mathbf{v}$ must be trivial.

(c) implies (a). This is obvious. $\square$

Proof of Theorem 4.1. Based on the results in Section 3.2, it is routine to verify that the identities Ⓢ hold in every semigroup of order three so that the variety $\mathbf{S}_3$ satisfies the identities Ⓢ. It remains to show that any identity $\mathbf{u} \approx \mathbf{v}$ of the variety $\mathbf{S}_3$ is a consequence of the identities Ⓢ. In the presence of Lemma 4.3, it suffices to assume that the identity $\mathbf{u} \approx \mathbf{v}$ is canonical. The identity $\mathbf{u} \approx \mathbf{v}$ is then trivial by Lemma 4.5 and so is vacuously a consequence of the identities Ⓢ. □

Corollary 4.6. *The variety $\mathbf{S}_3$ is defined by the identities*

$$x^8 y \approx x^2 y, \qquad xy^8 \approx xy^2, \qquad x^7 yx \approx xyx, \tag{4.5}$$

$$x^2 yx \approx xyx^2, \qquad xyxzx \approx x^2 yzx, \tag{4.6}$$

$$xyxy \approx x^2 y^2. \tag{4.7}$$

Proof. It is obvious that the deductions $\{(4.1), (4.2), (4.3)\} \vdash \{(4.5), (4.6), (4.7)\}$ and $\{(4.5), (4.6), (4.7)\} \vdash \{(4.1), (4.2)\}$ hold. Since

$$\begin{aligned} xh^m xyt^n y &\overset{(4.5)}{\approx} x^7 h^m xy^7 t^n y \overset{(4.6)}{\approx} x^6 h^m x^2 y^7 t^n y \overset{(4.7)}{\approx} \left(x^6 h^m xyx\right) y^6 t^n y \\ &\overset{(4.6)}{\approx} x^7 h^m \left(yxy^6 t^n y\right) \overset{(4.6)}{\approx} x^7 h^m y^7 xt^n y \overset{(4.5)}{\approx} xh^m yxt^n y \end{aligned}$$

for any $m, n \in \{0, 1\}$, the deduction $\{(4.5), (4.6), (4.7)\} \vdash (4.3)$ also holds. □

5. On Theorem 1.1

Lemma A. *Let $\mathbf{V}$ be any subvariety of $\mathbf{S}_3$. Then*

$$\mathbf{V} = \mathbf{S}_3\{\Lambda, \Pi\}$$

for some $\Lambda \subseteq \Lambda_7$ and some set Π of 6-balanced canonical identities.

Lemma B. *Let $\mathbf{u} \approx \mathbf{v}$ be any 6-balanced canonical identity that is* C*-compliant. Then*

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\left\{\Pi, \mathbf{u}' \approx \mathbf{v}'\right\}$$

for some set Π of 6-balanced canonical identities that are ip–fp*-compliant and some balanced canonical identity $\mathbf{u}' \approx \mathbf{v}'$.*

Lemma C. *Let $\mathbf{u} \approx \mathbf{v}$ be any 6-balanced canonical identity that is* ip–fp*-compliant. Then*

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\Lambda\}$$

for some $\Lambda \subseteq \Lambda_8$.

Lemma D. *Let $\mathbf{u} \approx \mathbf{v}$ be any balanced identity that is not a permutation identity. Then*

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\left\{\Lambda, \mathbf{u}' \approx \mathbf{v}'\right\}$$

for some $\Lambda \subseteq \Lambda_3$ and some identity $\mathbf{u}' \approx \mathbf{v}'$ that is ip–fp*-compliant.*

Lemmas A–D will be verified in Sections 6–9 respectively.

Proof of Theorem 1.1. Let $\mathbf{V}$ be any subvariety of $\mathbf{S}_3$. Then by Lemma A,

$$\mathbf{V} = \mathbf{S}_3\{\Lambda, \Pi\} \tag{5.1}$$

for some $\Lambda \subseteq \Lambda_7$ and some set Π of 6-balanced canonical identities.

Case 1. $L_2^1, R_2^1 \in \mathbf{V}$. By Lemma 3.2, parts (i) and (ii), the identities in Π are ip–fp-compliant. Therefore by Lemma C, there exists some $\Lambda' \subseteq \Lambda_8$ such that $\mathbf{S}_3\{\Pi\} = \mathbf{S}_3\{\Lambda'\}$. The variety $\mathbf{V} = \mathbf{S}_3\{\Lambda, \Lambda'\}$ is then finitely based by Lemma 2.2.

Case 2. $Y_2 \in \mathbf{V}$. By Lemma 3.2(iii), the identities in Π are C-compliant. It follows from Lemma B that

$$\mathbf{S}_3\{\Pi\} = \mathbf{S}_3\big\{\Pi', \Pi''\big\} \tag{5.2}$$

for some set Π' of 6-balanced canonical identities that are ip–fp-compliant and some set Π'' of balanced canonical identities. If the set Π'' contains some permutation identity, then the variety $\mathbf{V}$ is finitely based by Lemma 2.1. Therefore assume that the set Π'' does not contain any permutation identity, whence it follows from Lemma D that

$$\mathbf{S}_3\big\{\Pi''\big\} = \mathbf{S}_3\big\{\Lambda', \Pi'''\big\} \tag{5.3}$$

for some $\Lambda' \subseteq \Lambda_3$ and some set Π''' of identities that are ip–fp-compliant. Therefore

$$\mathbf{V} = \mathbf{S}_3\big\{\Lambda, \Lambda'\big\} \cap \mathbf{S}_3\big\{\Pi', \Pi'''\big\}$$

by (5.1), (5.2), and (5.3). Now the identities in $\Pi' \cup \Pi'''$ are ip–fp-compliant so that by Lemma 3.2, parts (i) and (ii), the variety $\mathbf{S}_3\{\Pi', \Pi'''\}$ contains the monoids L_2^1 and R_2^1. Therefore the variety $\mathbf{S}_3\{\Pi', \Pi'''\}$ is finitely based by Case 1. The variety $\mathbf{S}_3\{\Lambda, \Lambda'\}$ is finitely based by Lemma 2.2. Consequently, the variety $\mathbf{V}$ is also finitely based.

Case 3. $Y_2 \notin \mathbf{V}$. Then it follows from [11] that $\mathbf{V} = \mathbf{S}_3\{\Sigma, \sigma\}$ for some set Σ of C-compliant identities and some non-C-compliant identity σ. By Lemma 3.2(iii), the subvariety $\mathbf{S}_3\{\Sigma\}$ of $\mathbf{S}_3$ contains the semilattice Y_2 and so is finitely based by Case 2. Consequently, the variety $\mathbf{V} = \mathbf{S}_3\{\Sigma\} \cap \mathbf{S}_3\{\sigma\}$ is also finitely based. □

6. On Lemma A

For any identity $\mathbf{u} \approx \mathbf{v}$, denote by $\mathsf{B}_n(\mathbf{u} \approx \mathbf{v})$ the set of all letters of $\mathbf{u} \approx \mathbf{v}$ at which it is n-balanced, that is,

$$\mathsf{B}_n(\mathbf{u} \approx \mathbf{v}) = \big\{x \in \mathsf{C}(\mathbf{uv}) \bigm| \mathsf{m}(x, \mathbf{u}) \equiv \mathsf{m}(x, \mathbf{v}) \pmod n\big\}.$$

Clearly, an identity $\mathbf{u} \approx \mathbf{v}$ is n-balanced if and only if $\mathsf{B}_n(\mathbf{u} \approx \mathbf{v}) = \mathsf{C}(\mathbf{uv})$.

The identities from Λ_7 that are required for Lemma A are

$$x^5 \approx x^3, \tag{6.1}$$

$$x^6 \approx x^3, \tag{6.2}$$

$$x^8 \approx x^2. \tag{6.3}$$

Lemma 6.1. *Let $\mathbf{u} \approx \mathbf{v}$ be any canonical identity that is non-6-balanced at the letter x_1. Then*

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\Lambda^{(1)}, \Pi^{(1)}\}$$

for some set $\Lambda^{(1)}$ of identities from $\{(6.1), (6.2), (6.3)\}$ and some set $\Pi^{(1)}$ of identities $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ that satisfy the following conditions:

(a) $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ *is a canonical identity,*
(b) $\mathsf{C}(\mathbf{u}^{(1)}\mathbf{v}^{(1)}) = \mathsf{C}(\mathbf{uv})$,
(c) $\mathsf{B}_6(\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}) = \mathsf{B}_6(\mathbf{u} \approx \mathbf{v}) \cup \{x_1\}$.

Proof. Without loss of generality, assume that $\mathsf{m}(x_1, \mathbf{u}) = e + 1 > \mathsf{m}(x_1, \mathbf{v})$ and $\mathbf{u} = \mathbf{u}_1 x_1^e \mathbf{u}_2 x_1 \mathbf{u}_3$ for some $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in (\mathcal{X}\backslash\{x\})^*$. Let φ_1 be the substitution $p \mapsto x_1^6$ for all $p \in \mathcal{X}\backslash\{x_1\}$. There are three cases depending on the value of the difference $\mathsf{m}(x_1, \mathbf{u}) - \mathsf{m}(x_1, \mathbf{v})$ modulo 6.

Case 1. $\mathsf{m}(x_1, \mathbf{u}) - \mathsf{m}(x_1, \mathbf{v}) \equiv d \pmod 6$ with $d \in \{2, 4\}$. Choose any integer $r \geqslant 0$ such that $r + \mathsf{m}(x_1, \mathbf{v}) \equiv 3 \pmod 6$. Then

$$x_1^{6+r}(\mathbf{u}\varphi_1) \overset{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{u})} \overset{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{v})+d} \overset{(4.1)}{\approx} x_1^{3+d},$$
$$x_1^{6+r}(\mathbf{v}\varphi_1) \overset{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{v})} \overset{(4.1)}{\approx} x_1^{3},$$

whence

$$\mathbf{u} \approx \mathbf{v} \Vdash x^3 \approx x^{3+d} \approx x^{3+2d} \approx \cdots \approx x^{3+8} \overset{(4.1)}{\approx} x^5 \vdash (6.1).$$

Therefore

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{(6.1), \mathbf{u} \approx \mathbf{v}\}. \tag{6.4}$$

By the assumption of this case, it is necessary that $\mathsf{m}(x_1, \mathbf{u}) \geqslant 2$. There are two subcases.

1.1. $\mathsf{m}(x_1, \mathbf{u}) > 2$. Then $e \geqslant 2$. Let $\mathbf{v}^{(1)} = \mathbf{v}$. By Lemma 4.3, there exists some word $\mathbf{u}^{(1)}$ in canonical form such that $\mathbf{u}^{(1)} \overset{Ⓢ}{\approx} \mathbf{u}_1 x_1^{e+6-d} \mathbf{u}_2 x_1 \mathbf{u}_3$. Since

$$\mathbf{u}^{(1)} \overset{Ⓢ}{\approx} \mathbf{u}_1 x_1^{e+6-d} \mathbf{u}_2 x_1 \mathbf{u}_3 \overset{(6.1)}{\approx} \mathbf{u}_1 x_1^{e+6} \mathbf{u}_2 x_1 \mathbf{u}_3 \overset{(4.1)}{\approx} \mathbf{u},$$

it follows from (6.4) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{(6.1), \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}$. Conditions (a) and (b) clearly hold. Condition (c) also holds because

$$\mathsf{m}(x_1, \mathbf{u}^{(1)}) = \mathsf{m}(x_1, \mathbf{u}) + 6 - d \equiv \mathsf{m}(x_1, \mathbf{v}) = \mathsf{m}(x_1, \mathbf{v}^{(1)}) \pmod 6.$$

1.2. $\mathsf{m}(x_1, \mathbf{u}) = 2$. Then $\mathsf{m}(x_1, \mathbf{v}) = 0$ so that $x_1 \notin \mathsf{C}(\mathbf{v})$.

1.2.1. $|\mathsf{C}(\mathbf{u})| = 1$. Then $\mathbf{u} = x_1^2$. Let $\mathbf{u}^{(1)} = x_1^6$ and $\mathbf{v}^{(1)} = \mathbf{v}$. Let φ_2 be the substitution $x_1 \mapsto x_1^3$ so that $\mathbf{u}\varphi_2 = x_1^6$ and $\mathbf{v}\varphi_2 = \mathbf{v}$. Since

$$\begin{aligned}\{(6.1), \mathbf{u} \approx \mathbf{v}\} &\vdash x_1^2 = \mathbf{u} \approx \mathbf{v} = \mathbf{v}\varphi_2 \approx \mathbf{u}\varphi_2 \approx x_1^6 \stackrel{(6.1)}{\approx} x_1^8 \\ &\vdash \{(6.3), \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}, \\ \{(6.1), (6.3), \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\} &\vdash \mathbf{u} \stackrel{(6.3)}{\approx} x_1^8 \stackrel{(6.1)}{\approx} x_1^6 = \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)} = \mathbf{v} \\ &\vdash \mathbf{u} \approx \mathbf{v},\end{aligned}$$

it follows from (6.4) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{(6.1), (6.3), \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}$. It is easy to verify that the identity $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ satisfies conditions (a), (b), and (c).

1.2.2. $|\mathsf{C}(\mathbf{u})| \geqslant 2$. Then $\mathbf{u} = \mathbf{u}_1 x_1 \mathbf{u}_2 x_1 \mathbf{u}_3$. Let $\mathbf{u}^{(1)} = \mathbf{u}_1 x_1^5 \mathbf{u}_2 x_1 \mathbf{u}_3$ and $\mathbf{v}^{(1)} = \mathbf{v}$. Since

$$\mathbf{u} \stackrel{(4.1)}{\approx} \mathbf{u}_1 x_1^7 \mathbf{u}_2 x_1 \mathbf{u}_3 \stackrel{(6.1)}{\approx} \mathbf{u}_1 x_1^5 \mathbf{u}_2 x_1 \mathbf{u}_3 = \mathbf{u}^{(1)},$$

it follows from (6.4) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{(6.1), \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}$ where the identity $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ satisfies conditions (a), (b), and (c)

Case 2. $\mathsf{m}(x_1, \mathbf{u}) - \mathsf{m}(x_1, \mathbf{v}) \equiv 3 \pmod 6$. Choose any integer $r \geqslant 0$ such that $r + \mathsf{m}(x_1, \mathbf{v}) \equiv 0 \pmod 6$. Then

$$\begin{aligned} x_1^{6+r}(\mathbf{u}\varphi_1) &\stackrel{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{u})} \stackrel{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{v})+3} \stackrel{(4.1)}{\approx} x_1^3, \\ x_1^{6+r}(\mathbf{v}\varphi_1) &\stackrel{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{v})} \stackrel{(4.1)}{\approx} x_1^6, \end{aligned}$$

whence the deduction $\mathbf{u} \approx \mathbf{v} \Vdash (6.2)$ holds. Thus $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{(6.2), \mathbf{u} \approx \mathbf{v}\}$. Let $\mathbf{v}^{(1)} = \mathbf{v}$. By Lemma 4.3, there exists some word $\mathbf{u}^{(1)}$ in canonical form such that $\mathbf{u}^{(1)} \stackrel{Ⓢ}{\approx} \mathbf{u}_1 x_1^{e+3} \mathbf{u}_2 x_1 \mathbf{u}_3$. Then $\mathbf{S}_3\{(6.2), \mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{(6.2), \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}$ since

$$\mathbf{u}^{(1)} \stackrel{Ⓢ}{\approx} \mathbf{u}_1 x_1^{e+3} \mathbf{u}_2 x_1 \mathbf{u}_3 \stackrel{(6.2)}{\approx} \mathbf{u}_1 x_1^{e+6} \mathbf{u}_2 x_1 \mathbf{u}_3 \stackrel{(4.1)}{\approx} \mathbf{u}.$$

Hence $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{(6.2), \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}$. It is clear that conditions (a) and (b) hold. Since

$$\mathsf{m}(x_1, \mathbf{u}^{(1)}) \equiv \mathsf{m}(x_1, \mathbf{u}) + 3 \equiv \mathsf{m}(x_1, \mathbf{v}) = \mathsf{m}(x_1, \mathbf{v}^{(1)}) \pmod 6,$$

condition (c) also holds.

Case 3. $\mathsf{m}(x_1, \mathbf{u}) - \mathsf{m}(x_1, \mathbf{v}) \equiv d \pmod 6$ with $d \in \{1, 5\}$. Choose any integer $r \geqslant 0$ such that $r + \mathsf{m}(x_1, \mathbf{v}) \equiv 4 \pmod 6$. Then

$$\begin{aligned} x_1^{6+r}(\mathbf{u}\varphi_1) &\stackrel{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{u})} \stackrel{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{v})+d} \stackrel{(4.1)}{\approx} x_1^{4+d}, \\ x_1^{6+r}(\mathbf{v}\varphi_1) &\stackrel{(4.1)}{\approx} x_1^{6+r+\mathsf{m}(x_1,\mathbf{v})} \stackrel{(4.1)}{\approx} x_1^4, \end{aligned}$$

whence

$$\mathbf{u} \approx \mathbf{v} \Vdash x^4 \approx x^{4+d} \approx x^{4+5} \stackrel{(4.1)}{\approx} x^3 \vdash x^4 \approx x^3 \vdash \{(6.1), (6.2)\}.$$

Therefore

$$\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\big\{(6.1),(6.2),\mathbf{u}\approx\mathbf{v}\big\}. \tag{6.5}$$

By the assumption of this case, it is necessary that $\mathsf{m}(x_1,\mathbf{u})\geqslant 1$. There are four subcases.

3.1. $\mathsf{m}(x_1,\mathbf{u})>2$. Then $e\geqslant 2$. Let $\mathbf{v}^{(1)}=\mathbf{v}$. By Lemma 4.3, there exists some word $\mathbf{u}^{(1)}$ in canonical form such that $\mathbf{u}^{(1)}\overset{\circledS}{\approx}\mathbf{u}_1x^{e+6-d}\mathbf{u}_2x\mathbf{u}_3$. Since

$$\mathbf{u}^{(1)}\overset{\circledS}{\approx}\mathbf{u}_1x_1^{e+6-d}\mathbf{u}_2x_1\mathbf{u}_3\overset{(6.1)}{\approx}\mathbf{u}_1x_1^{e+9}\mathbf{u}_2x_1\mathbf{u}_3\overset{(6.2)}{\approx}\mathbf{u}_1x_1^{e+6}\mathbf{u}_2x_1\mathbf{u}_3\overset{(4.1)}{\approx}\mathbf{u},$$

it follows from (6.5) that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{(6.1),(6.2),\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}\}$. Similar to Subcase 1.1, conditions (a), (b), and (c) hold.

3.2. $\mathsf{m}(x_1,\mathbf{u})=2$ and $|\mathsf{C}(\mathbf{u})|\geqslant 2$. Then $\mathbf{u}=\mathbf{u}_1x_1\mathbf{u}_2x_1\mathbf{u}_3$ and $\mathsf{m}(x_1,\mathbf{v})=1$. Let $\mathbf{u}^{(1)}=\mathbf{u}_1x_1^6\mathbf{u}_2x_1\mathbf{u}_3$ and $\mathbf{v}^{(1)}=\mathbf{v}$. Since

$$\mathbf{u}\overset{(4.1)}{\approx}\mathbf{u}_1x_1^7\mathbf{u}_2x_1\mathbf{u}_3\overset{(6.2)}{\approx}\mathbf{u}_1x_1^4\mathbf{u}_2x_1\mathbf{u}_3\overset{(6.1)}{\approx}\mathbf{u}_1x_1^6\mathbf{u}_2x_1\mathbf{u}_3=\mathbf{u}^{(1)},$$

it follows from (6.5) that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{(6.1),(6.2),\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}\}$ where the identity $\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}$ satisfies conditions (a), (b), and (c).

3.3. $\mathsf{m}(x_1,\mathbf{u})=2$ and $|\mathsf{C}(\mathbf{u})|=1$. Then $\mathbf{u}=x_1^2$ and $\mathsf{m}(x_1,\mathbf{v})=1$.

3.3.1. $\mathsf{C}(\mathbf{u})=\mathsf{C}(\mathbf{v})$. Then the identity $\mathbf{u}\approx\mathbf{v}$ is $x_1^2\approx x_1$ and so it is easy to show by (6.5) that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{(6.1),(6.2),x_1^7\approx x_1\}$ where the identity $x_1^7\approx x_1$ satisfies conditions (a), (b), and (c).

3.3.2. $\mathsf{C}(\mathbf{u})\neq\mathsf{C}(\mathbf{v})$. Let $\mathbf{u}^{(1)}=x_1^7$ and $\mathbf{v}^{(1)}=\mathbf{v}$. Since

$$\begin{aligned}&\big\{(6.1),(6.2),\mathbf{u}\approx\mathbf{v}\big\}\\ &\quad\Vdash\left\{\begin{matrix}x_1^2=\mathbf{u}=\mathbf{u}\varphi_1\approx\mathbf{v}\varphi_1\overset{(4.1)}{\approx}x_1^7\overset{(6.1)}{\approx}x_1^5\overset{(6.2)}{\approx}x_1^8\\ \mathbf{u}^{(1)}=x_1^7\overset{(4.1)}{\approx}\mathbf{v}\varphi_1\approx\mathbf{u}\varphi_1=\mathbf{u}\approx\mathbf{v}=\mathbf{v}^{(1)}\end{matrix}\right\}\\ &\quad\vdash\big\{(6.3),\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}\big\}\end{aligned}$$

and

$$\begin{aligned}&\big\{(6.1),(6.2),(6.3),\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}\big\}\\ &\quad\vdash\mathbf{u}=x_1^2\overset{(6.3)}{\approx}x_1^8\overset{(6.2)}{\approx}x_1^5\overset{(6.1)}{\approx}x_1^7=\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}=\mathbf{v}\\ &\quad\vdash\mathbf{u}\approx\mathbf{v},\end{aligned}$$

it follows from (6.5) that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{(6.1),(6.2),(6.3),\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}\}$. It is easy to show that the identity $\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}$ satisfies conditions (a), (b), and (c).

3.4. $\mathsf{m}(x_1,\mathbf{u})=1$. Then $e=0$ and $x_1\notin\mathsf{C}(\mathbf{v})$ so that $\mathbf{u}=\mathbf{u}_1\mathbf{u}_2x_1\mathbf{u}_3$. The identities in $\Pi^{(1)}=\{\mathbf{u}\approx\mathbf{u}_1\mathbf{u}_2x_1^7\mathbf{u}_3,\mathbf{v}\approx\mathbf{u}_1\mathbf{u}_2x_1^6\mathbf{u}_3\}$ satisfy conditions (a), (b), and (c). Let φ_3 be the substitution $x_1\mapsto x_1^6$. Since

$$\begin{aligned}\big\{(6.1),(6.2),\mathbf{u}\approx\mathbf{v}\big\}\Vdash\mathbf{u}&\approx\mathbf{v}=\mathbf{v}\varphi_3\approx\mathbf{u}\varphi_3=\mathbf{u}_1\mathbf{u}_2x_1^6\mathbf{u}_3\\ &\overset{(6.1)}{\approx}\mathbf{u}_1\mathbf{u}_2x_1^4\mathbf{u}_3\overset{(6.2)}{\approx}\mathbf{u}_1\mathbf{u}_2x_1^7\mathbf{u}_3\\ &\vdash\Pi^{(1)},\end{aligned}$$

$$\begin{aligned}\big\{(6.1),(6.2)\big\}\cup\Pi^{(1)}\Vdash \mathbf{u} \ &\approx \ \mathbf{u}_1\mathbf{u}_2x_1^7\mathbf{u}_3 \overset{(6.2)}{\approx} \mathbf{u}_1\mathbf{u}_2x_1^4\mathbf{u}_3 \\ &\overset{(6.1)}{\approx} \mathbf{u}_1\mathbf{u}_2x_1^6\mathbf{u}_3\approx\mathbf{v} \\ \vdash \mathbf{u} \ &\approx \ \mathbf{v},\end{aligned}$$

it follows from (6.5) that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{(6.1),(6.2),\Pi^{(1)}\}$. □

Lemma 6.2. *Let $\mathbf{u}\approx\mathbf{v}$ be any canonical identity. Then*

$$\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{\Lambda,\Pi\}$$

for some set Λ of identities from $\{(6.1),(6.2),(6.3)\}$ and some set Π of 6-balanced canonical identities.

Proof. Suppose that $\mathbf{u}\approx\mathbf{v}$ is non-6-balanced precisely at the letters $x_1,\dots,x_m$. It follows from Lemma 6.1 that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{\Lambda^{(1)},\Pi^{(1)}\}$ for some subset $\Lambda^{(1)}$ of $\{(6.1),(6.2),(6.3)\}$ and some set $\Pi^{(1)}$ of canonical identities $\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}$ such that $\mathsf{C}(\mathbf{u}^{(1)}\mathbf{v}^{(1)})=\mathsf{C}(\mathbf{uv})$ and $\mathsf{B}_6(\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)})=\mathsf{B}_6(\mathbf{u}\approx\mathbf{v})\cup\{x_1\}$. Note that each identity in $\Pi^{(1)}$ involves the same letters as $\mathbf{u}\approx\mathbf{v}$ and is non-6-balanced precisely at the letters $x_2,\dots,x_m$. Therefore Lemma 6.1 can be applied to every identity in $\Pi^{(1)}$ resulting in $\mathbf{S}_3\{\Pi^{(1)}\}=\mathbf{S}_3\{\Lambda^{(2)},\Pi^{(2)}\}$ for some subset $\Lambda^{(2)}$ of $\{(6.1),(6.2),(6.3)\}$ and some set $\Pi^{(2)}$ of canonical identities $\mathbf{u}^{(2)}\approx\mathbf{v}^{(2)}$ such that $\mathsf{C}(\mathbf{u}^{(2)}\mathbf{v}^{(2)})=\mathsf{C}(\mathbf{u}^{(1)}\mathbf{v}^{(1)})=\mathsf{C}(\mathbf{uv})$ and

$$\mathsf{B}_6\big(\mathbf{u}^{(2)}\approx\mathbf{v}^{(2)}\big)=\mathsf{B}_6\big(\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}\big)\cup\{x_2\}=\mathsf{B}_6(\mathbf{u}\approx\mathbf{v})\cup\{x_1,x_2\}.$$

Now each identity in $\Pi^{(2)}$ involves the same letters as $\mathbf{u}\approx\mathbf{v}$ and is non-6-balanced precisely at the letters $x_3,\dots,x_m$. It is easy to see how the sets $\Lambda^{(i)}$ and $\Pi^{(i)}$ can be obtained for $3\leqslant i\leqslant m$. Let $\Lambda=\Lambda^{(1)}\cup\dots\cup\Lambda^{(m)}$ and $\Pi=\Pi^{(m)}$. Then Λ is some subset of $\{(6.1),(6.2),(6.3)\}$ and each identity $\mathbf{u}^{(m)}\approx\mathbf{v}^{(m)}$ in Π is 6-balanced since $\mathsf{C}(\mathbf{u}^{(m)}\mathbf{v}^{(m)})=\mathsf{C}(\mathbf{uv})$ and

$$\mathsf{B}_6\big(\mathbf{u}^{(m)}\approx\mathbf{v}^{(m)}\big)=\mathsf{B}_6(\mathbf{u}\approx\mathbf{v})\cup\{x_1,\dots,x_m\}.$$

Therefore

$$\begin{aligned}\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}&=\mathbf{S}_3\big\{\Lambda^{(1)},\Pi^{(1)}\big\}\\&=\mathbf{S}_3\big\{\Lambda^{(1)},\Lambda^{(2)},\Pi^{(2)}\big\}\\&\ \ \vdots\\&=\mathbf{S}_3\big\{\Lambda^{(1)},\Lambda^{(2)},\dots,\Lambda^{(m)},\Pi^{(m)}\big\}\\&=\mathbf{S}_3\{\Lambda,\Pi\}\end{aligned}$$

as required. □

Proof of Lemma A. It is easy to see that Lemma A follows from Lemmas 4.3 and 6.2. □

7. On Lemma B

For any identity $\mathbf{u} \approx \mathbf{v}$, denote by $\mathsf{B}(\mathbf{u} \approx \mathbf{v})$ the set of all letters of $\mathbf{u} \approx \mathbf{v}$ at which it is balanced, that is,

$$\mathsf{B}(\mathbf{u} \approx \mathbf{v}) = \{x \in \mathsf{C}(\mathbf{uv}) \mid \mathsf{m}(x, \mathbf{u}) = \mathsf{m}(x, \mathbf{v})\}.$$

Clearly, an identity $\mathbf{u} \approx \mathbf{v}$ is balanced if and only if $\mathsf{B}(\mathbf{u} \approx \mathbf{v}) = \mathsf{C}(\mathbf{uv})$.

Remark 7.1. Suppose that $\mathbf{u} \approx \mathbf{v}$ is any canonical 6-balanced identity such that $1 \leqslant \mathsf{m}(x, \mathbf{u}) < \mathsf{m}(x, \mathbf{v})$ for some $x \in \mathcal{X}$. Since $\mathsf{m}(x, \mathbf{u}) \equiv \mathsf{m}(x, \mathbf{v}) \pmod 6$, it follows from (CF2) and (CF3) that there can only be two possibilities:

(a) $(\mathsf{m}(x, \mathbf{u}), \mathsf{m}(x, \mathbf{v})) = (2, 8)$,
(b) $(\mathsf{m}(x, \mathbf{u}), \mathsf{m}(x, \mathbf{v})) = (1, 7)$.

Note that by (CF3), it is necessary that $\mathbf{v} = x^8$ in (a).

Lemma 7.2. *Let $\mathbf{u} \approx \mathbf{v}$ be any 6-balanced canonical identity that is C-compliant. Suppose that $\mathbf{u} \approx \mathbf{v}$ is non-balanced at the letter x_1. Then*

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}, \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}$$

for some 6-balanced canonical identity $\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}$ that is ip–fp-compliant and some 6-balanced canonical identity $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ such that $\mathsf{C}(\mathbf{u}^{(1)}\mathbf{v}^{(1)}) = \mathsf{C}(\mathbf{uv})$ and $\mathsf{B}(\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}) = \mathsf{B}(\mathbf{u} \approx \mathbf{v}) \cup \{x_1\}$.

Proof. Without loss of generality, assume that $1 \leqslant \mathsf{m}(x_1, \mathbf{u}) < \mathsf{m}(x_1, \mathbf{v})$. If the identity $\mathbf{u} \approx \mathbf{v}$ involves only one letter, then it can only be $x_1^2 \approx x_1^8$, whence the lemma holds by letting $\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}$ be $x_1^2 \approx x_1^8$ and $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ be $x_1 \approx x_1$. Therefore assume that the identity $\mathbf{u} \approx \mathbf{v}$ involves at least two distinct letters. Then $\mathsf{m}(x_1, \mathbf{u}) = 1$ and $\mathsf{m}(x_1, \mathbf{v}) = 7$ as observed in Remark 7.1. Let φ be the substitution $x_1 \mapsto x_1^7$. It is clear that the word $\mathbf{u}\varphi$ is in canonical form so that identities $\mathbf{u}\varphi \approx \mathbf{u}$ and $\mathbf{u}\varphi \approx \mathbf{v}$ are 6-balanced canonical identities. Since

$$\mathbf{u} \approx \mathbf{v} \Vdash \mathbf{u}\varphi \approx \mathbf{v}\varphi \overset{(4.1)}{\approx} \mathbf{v} \approx \mathbf{u},$$

it follows that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}\varphi \approx \mathbf{u}, \mathbf{u}\varphi \approx \mathbf{v}\}$. The lemma now holds by letting $\mathbf{u}\varphi \approx \mathbf{u}$ be $\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}$ and $\mathbf{u}\varphi \approx \mathbf{v}$ be $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$. □

Proof of Lemma B. Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ is non-balanced precisely at the letters $x_1, \ldots, x_m$. It follows from Lemma 7.2 that

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}, \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\}$$

for some identity $\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}$ that is ip–fp-compliant and some 6-balanced canonical identity $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ such that $\mathsf{C}(\mathbf{u}^{(1)}\mathbf{v}^{(1)}) = \mathsf{C}(\mathbf{uv})$ and

$$\mathsf{B}(\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}) = \mathsf{B}(\mathbf{u} \approx \mathbf{v}) \cup \{x_1\}.$$

For $1 < i \leqslant m$, the same lemma can be applied to obtain

$$\mathbf{S}_3\{\mathbf{u}^{(i-1)} \approx \mathbf{v}^{(i-1)}\} = \mathbf{S}_3\{\mathbf{p}^{(i)} \approx \mathbf{q}^{(i)}, \mathbf{u}^{(i)} \approx \mathbf{v}^{(i)}\}$$

for some identity $\mathbf{p}^{(i)} \approx \mathbf{q}^{(i)}$ that is ip–fp-compliant and some 6-balanced canonical identity $\mathbf{u}^{(i)} \approx \mathbf{v}^{(i)}$ such that $\mathsf{C}(\mathbf{u}^{(i)}\mathbf{v}^{(i)}) = \mathsf{C}(\mathbf{uv})$ and

$$\mathsf{B}(\mathbf{u}^{(i)} \approx \mathbf{v}^{(i)}) = \mathsf{B}(\mathbf{u} \approx \mathbf{v}) \cup \{x_1, \ldots, x_i\}.$$

Let $\Pi = \{\mathbf{p}^{(i)} \approx \mathbf{q}^{(i)} \mid 1 \leqslant i \leqslant m\}$ and let $\mathbf{u}' \approx \mathbf{v}'$ be $\mathbf{u}^{(m)} \approx \mathbf{v}^{(m)}$. Then

$$\begin{aligned}
\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} &= \mathbf{S}_3\{\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}, \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}\} \\
&= \mathbf{S}_3\{\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}, \mathbf{p}^{(2)} \approx \mathbf{q}^{(2)}, \mathbf{u}^{(2)} \approx \mathbf{v}^{(2)}\} \\
&\ \ \vdots \\
&= \mathbf{S}_3\{\mathbf{p}^{(1)} \approx \mathbf{q}^{(1)}, \ldots, \mathbf{p}^{(m)} \approx \mathbf{q}^{(m)}, \mathbf{u}^{(m)} \approx \mathbf{v}^{(m)}\} \\
&= \mathbf{S}_3\{\Pi, \mathbf{u}' \approx \mathbf{v}'\}.
\end{aligned}$$

The canonical identity $\mathbf{u}' \approx \mathbf{v}'$ is balanced since it satisfies the conditions that $\mathsf{C}(\mathbf{u}'\mathbf{v}') = \mathsf{C}(\mathbf{uv})$ and $\mathsf{B}(\mathbf{u}' \approx \mathbf{v}') = \mathsf{B}(\mathbf{u} \approx \mathbf{v}) \cup \{x_1, \ldots, x_m\}$. □

8. On Lemma C

The identities in Λ_8 that are required in Lemma C are

$$\left.\begin{array}{ll}
(8.1_\ell^r): & h_1 \cdots h_\ell x^7 t_1 \cdots t_r \approx h_1 \cdots h_\ell x t_1 \cdots t_r, \\
(8.1_\ell^\infty): & h_1 \cdots h_\ell x^7 t^2 \approx h_1 \cdots h_\ell x t^2, \\
(8.1_\infty^r): & h^2 x^7 t_1 \cdots t_r \approx h^2 x t_1 \cdots t_r, \\
(8.1_\infty^\infty): & h^2 x^7 t^2 \approx h^2 x t^2, \\
(8.1\dagger): & h x^7 h \approx h x h,
\end{array}\right\} \tag{8.1}$$

where $\ell, r \in \{0, 1, \ldots\}$.

Proof of Lemma C. Let $\mathbf{u} \approx \mathbf{v}$ be any 6-balanced canonical identity that is ip–fp-compliant. If $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbf{S}_3$, then the lemma clearly holds with $\Lambda = \emptyset$. Therefore assume that $\mathbf{u} \approx \mathbf{v}$ is not an identity of $\mathbf{S}_3$, whence it follows from Lemma 4.5 that $\mathbf{u} \approx \mathbf{v}$ is non-balanced, that is,

(a) $\mathsf{m}(x, \mathbf{u}) \equiv \mathsf{m}(x, \mathbf{v}) \pmod 6$ for all $x \in \mathcal{X}$, and
(b) $\mathsf{m}(x, \mathbf{u}) \neq \mathsf{m}(x, \mathbf{v})$ for some $x \in \mathcal{X}$.

If $|\mathsf{C}(\mathbf{u})| = 1$, then since $\mathbf{u} \approx \mathbf{v}$ is a canonical identity, it follows from (a), (b), and (CF2) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\}$ is either $\mathbf{S}_3\{x^7 \approx x\}$ or $\mathbf{S}_3\{x^8 \approx x^2\}$. Therefore assume that $|\mathsf{C}(\mathbf{u})| \geqslant 2$. It follows from Lemma 4.4 that $\mathsf{a}(x, \mathbf{u}) = \mathsf{a}(x, \mathbf{v})$ for any $x \in \mathcal{X}$ and the appearance sequences of $\mathbf{u}$ and $\mathbf{v}$ coincide. Let $x_1 \cdots x_m$ be the appearance sequence of $\mathbf{u}$ and $\mathbf{v}$ so that

$$\mathbf{u} = x_1^{e_1} \cdots x_m^{e_m} \quad \text{and} \quad \mathbf{v} = x_1^{f_1} \cdots x_m^{f_m}$$

for some $e_i, f_i \in \{1, 2, \ldots, 7\}$ with $x_i \neq x_{i+1}$. Note that by (a), (CF3), and (CF4),

(c) if $x_i = x_{i+k}$ for some $k > 1$ (that is, $\mathsf{a}(x_i, \mathbf{u}) = \mathsf{a}(x_i, \mathbf{v}) = 2$), then $e_i = f_i \in \{1, 2, \ldots, 6\}$ and $e_{i+k} = f_{i+k} = 1$, whence $\mathsf{m}(x_i, \mathbf{u}) = \mathsf{m}(x_i, \mathbf{v})$.

Let $\mathbf{u}^{(0)} = \mathbf{u}$ and $\mathbf{v}^{(0)} = \mathbf{v}$. Let ℓ be the least such that $\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}$ is non-balanced at x_ℓ. It follows from (c) that $x_i \neq x_\ell$ for all $i \neq \ell$, whence $(e_\ell, f_\ell) \in \{(1,7), (7,1)\}$ by (a). Therefore

$$\mathbf{u}^{(0)} = x_1^{e_1} \cdots x_{\ell-1}^{e_{\ell-1}} x_\ell^{e_\ell} x_{\ell+1}^{e_{\ell+1}} \cdots x_m^{e_m},$$
$$\mathbf{v}^{(0)} = x_1^{e_1} \cdots x_{\ell-1}^{e_{\ell-1}} x_\ell^{f_\ell} x_{\ell+1}^{f_{\ell+1}} \cdots x_m^{f_m}.$$

Let

$$\mathbf{u}^{(1)} = x_1^{e_1} \cdots x_{\ell-1}^{e_{\ell-1}} x_\ell^{d} x_{\ell+1}^{e_{\ell+1}} \cdots x_m^{e_m},$$
$$\mathbf{v}^{(1)} = x_1^{e_1} \cdots x_{\ell-1}^{e_{\ell-1}} x_\ell^{d} x_{\ell+1}^{f_{\ell+1}} \cdots x_m^{f_m},$$

where either $d = 1$ or $d = 7$. It is first shown that

$$\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, \Lambda^{(1)}\}$$

for some subset $\Lambda^{(1)}$ of (8.1). There are two cases to consider depending on whether or not the sets $\{x_1, \ldots, x_{\ell-1}\}$ and $\{x_{\ell+1}, \ldots, x_m\}$ are disjoint.

Case 1. $\{x_1, \ldots, x_{\ell-1}\} \cap \{x_{\ell+1}, \ldots, x_m\} \neq \emptyset$. Then $x_i = x_j$ for some i, j with $i < \ell < j$. Let φ_1 be the substitution $x_k \mapsto z^6$ for all $k \neq \ell$. Then

$$\begin{aligned} \mathbf{u}^{(0)} \approx \mathbf{v}^{(0)} &\Vdash z x_\ell^{e_\ell} z \overset{(4.1)}{\approx} z\big(\mathbf{u}^{(0)}\varphi_1\big)z \approx z\big(\mathbf{v}^{(0)}\varphi_1\big)z \overset{(4.1)}{\approx} z x_\ell^{f_\ell} z \\ &\vdash (8.1\dagger) \end{aligned}$$

so that $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}, (8.1\dagger)\}$. If $e_\ell = 7$, then the deduction $(8.1\dagger) \Vdash \mathbf{u}^{(0)} \approx \mathbf{u}^{(1)}$ holds with $d = 1$ since

$$\begin{aligned} \mathbf{u}^{(0)} &= \cdots x_i \cdots x_\ell^7 \cdots x_j \cdots \\ &\overset{(4.1)}{\approx} \cdots x_i \cdots x_i^6 x_\ell^7 x_i^6 \cdots x_j \cdots \\ &\overset{(8.1\dagger)}{\approx} \cdots x_i \cdots x_i^6 x_\ell x_i^6 \cdots x_j \cdots \\ &\overset{(4.1)}{\approx} \cdots x_i \cdots x_\ell \cdots x_j \cdots \\ &= \mathbf{u}^{(1)}. \end{aligned}$$

Similarly, if $f_\ell = 7$, then the deduction $(8.1\dagger) \Vdash \mathbf{v}^{(0)} \approx \mathbf{v}^{(1)}$ holds with $d = 1$. Hence $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, (8.1\dagger)\}$ with $d = 1$.

Case 2. $\{x_1, \ldots, x_{\ell-1}\} \cap \{x_{\ell+1}, \ldots, x_m\} = \emptyset$.

2.1. $e_i = 1$ for all $i < \ell$ and $(e_j, f_j) = (1,1)$ for all $j > \ell$. Then the identity $\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}$ is $(8.1_{\ell-1}^{m-\ell})$ so that $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, (8.1_{\ell-1}^{m-\ell})\}$ (where $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ is a trivial identity regardless of the value of d).

2.2. $e_i = 1$ for all $i < \ell$ and $(e_j, f_j) \neq (1,1)$ for some $j > \ell$. Let φ_2 be the substitution $x_j \mapsto y^6$ for all $j > \ell$. Then

$$\begin{aligned}(4.1) &\vdash x_1 \cdots x_{\ell-1} x_\ell^{e_\ell} y^2 \overset{(4.1)}{\approx} \big(\mathbf{u}^{(0)}\varphi_2\big) y^2 \approx \big(\mathbf{v}^{(0)}\varphi_2\big) y^2 \overset{(4.1)}{\approx} x_1 \cdots x_{\ell-1} x_\ell^{f_\ell} y^2 \\ &\vdash \big(8.1_{\ell-1}^{\infty}\big)\end{aligned}$$

so that $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}, (8.1_{\ell-1}^{\infty})\}$. By symmetry, it suffices to assume that $e_j \neq 1$ for some $j > \ell$. Then the identity $(8.1_{\ell-1}^{\infty})$ can be used to convert the exponent e_ℓ of $\mathbf{u}^{(0)}$ from one to seven or from seven to one, that is,

$$\begin{aligned}\mathbf{u}^{(0)} &= x_1 \cdots x_{\ell-1} x_\ell^{e_\ell} x_{\ell+1}^{e_{\ell+1}} \cdots x_{j-1}^{e_{j-1}} x_j^{e_j} \cdots \\ &\overset{(8.1_{\ell-1}^{\infty})}{\approx} x_1 \cdots x_{\ell-1} x_\ell^{e_\ell} \big(x_{\ell+1}^{e_{\ell+1}} \cdots x_{j-1}^{e_{j-1}}\big)^7 x_j^{e_j} \cdots \\ &\overset{(8.1_{\ell-1}^{\infty})}{\approx} x_1 \cdots x_{\ell-1} x_\ell^{d} \big(x_{\ell+1}^{e_{\ell+1}} \cdots x_{j-1}^{e_{j-1}}\big)^7 x_j^{e_j} \cdots \\ &\overset{(8.1_{\ell-1}^{\infty})}{\approx} x_1 \cdots x_{\ell-1} x_\ell^{d} x_{\ell+1}^{e_{\ell+1}} \cdots x_{j-1}^{e_{j-1}} x_j^{e_j} \cdots\end{aligned}$$

where $d = f_\ell \in \{1, 7\}$. Thus the identity $(8.1_{\ell-1}^{\infty})$ can be used to convert the identity $\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}$ into $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$. Hence $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, (8.1_{\ell-1}^{\infty})\}$ for some $d \in \{1, 7\}$.

2.3. $e_i \geqslant 2$ for some $i < \ell$ and $(e_j, f_j) = (1, 1)$ for all $j > \ell$. This is symmetrical to Subcase 2.2. Hence $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, (8.1_{\infty}^{m-\ell})\}$ for some $d \in \{1, 7\}$.

2.4. $e_i \geqslant 2$ for some $i < \ell$ and $(e_j, f_j) \neq (1, 1)$ for some $j > \ell$. Let φ_3 be the substitution

$$x_i \mapsto \begin{cases} y^6 & \text{if } i < \ell, \\ z^6 & \text{if } i > \ell. \end{cases}$$

Then

$$\begin{aligned}(4.1) &\vdash y^2 x_\ell^{e_\ell} z^2 \overset{(4.1)}{\approx} y^2\big(\mathbf{u}^{(0)}\varphi_3\big) z^2 \approx y^2\big(\mathbf{v}^{(0)}\varphi_3\big) z^2 \overset{(4.1)}{\approx} y^2 x_\ell^{f_\ell} z^2 \\ &\vdash \big(8.1_{\infty}^{\infty}\big)\end{aligned}$$

so that $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}, (8.1_{\infty}^{\infty})\}$. By symmetry, it suffices to assume that $e_j \geqslant 2$ for some $j > \ell$. By assumption, $e_i \geqslant 2$ for some $i < \ell$. Then the identities $\{(4.1), (8.1_{\infty}^{\infty})\}$ can be used to convert the exponent e_ℓ of $\mathbf{u}^{(0)}$ from one to seven or from seven to one, that is,

$$\begin{aligned}\mathbf{u}^{(0)} &= \cdots x_i^{e_i} \cdots x_\ell^{e_\ell} \cdots x_j^{e_j} \cdots \\ &\overset{(8.1_{\infty}^{\infty})}{\approx} \cdots x_i^{e_i} \big(\cdots x_\ell^{e_\ell} \cdots\big)^7 x_j^{e_j} \cdots \\ &\overset{(4.1)}{\approx} \cdots x_i^{e_i} \big(\cdots x_\ell^{d} \cdots\big)^7 x_j^{e_j} \cdots \\ &\overset{(8.1_{\infty}^{\infty})}{\approx} \cdots x_i^{e_i} \cdots x_\ell^{d} \cdots x_j^{e_j} \cdots\end{aligned}$$

where $d = f_\ell \in \{1, 7\}$. Thus the identities $\{(4.1), (8.1_{\infty}^{\infty})\}$ can be used to convert $\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}$ into $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$, whence $\mathbf{S}_3\{\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}\} = \mathbf{S}_3\{\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, (8.1_{\infty}^{\infty})\}$ for some $d \in \{1, 7\}$.

Now note that if ℓ_1 is the least such that $\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}$ is non-balanced at x_{ℓ_1}, then $\ell < \ell_1$. For any $i \geqslant 1$, if the identity $\mathbf{u}^{(i)} \approx \mathbf{v}^{(i)}$ is nontrivial, then the procedure in Cases 1 and 2 can be repeated to construct an identity $\mathbf{u}^{(i+1)} \approx \mathbf{v}^{(i+1)}$ and some subset $\Lambda^{(i+1)}$ of (8.1) such that

- if ℓ_i is the least such that $\mathbf{u}^{(i)} \approx \mathbf{v}^{(i)}$ is non-balanced at x_{ℓ_i} and ℓ_{i+1} is the least such that $\mathbf{u}^{(i+1)} \approx \mathbf{v}^{(i+1)}$ is non-balanced at $x_{\ell_{i+1}}$, then $\ell_i < \ell_{i+1}$;
- $\mathbf{S}_3\{\mathbf{u}^{(i)} \approx \mathbf{v}^{(i)}\} = \mathbf{S}_3\{\mathbf{u}^{(i+1)} \approx \mathbf{v}^{(i+1)}, \Lambda^{(i+1)}\}$.

The construction of $\mathbf{u}^{(0)} \approx \mathbf{v}^{(0)}, \mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, \ldots$ and $\Lambda^{(1)}, \Lambda^{(2)}, \ldots$ cannot continue indefinitely since the indices $\ell_1, \ell_2, \ldots$ form an increasing sequence that is bounded above by m. This implies that the identity $\mathbf{u}^{(k)} \approx \mathbf{v}^{(k)}$ must be trivial for some sufficiently large k. Let $\Lambda = \Lambda^{(1)} \cup \cdots \cup \Lambda^{(k)}$. Then

$$\begin{aligned}
\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} &= \mathbf{S}_3\big\{\mathbf{u}^{(1)} \approx \mathbf{v}^{(1)}, \Lambda^{(1)}\big\} \\
&= \mathbf{S}_3\big\{\mathbf{u}^{(2)} \approx \mathbf{v}^{(2)}, \Lambda^{(1)}, \Lambda^{(2)}\big\} \\
&\;\;\vdots \\
&= \mathbf{S}_3\big\{\mathbf{u}^{(k)} \approx \mathbf{v}^{(k)}, \Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(k)}\big\} \\
&= \mathbf{S}_3\{\Lambda\}
\end{aligned}$$

as required. □

9. On Lemma D

The identities from Λ_3 that are required in Lemma D are

$$xhyztx \approx xhzytx, \qquad xhyzx \approx xhzyx, \qquad xyztx \approx xzytx, \qquad xyzx \approx xzyx, \tag{9.1}$$

and

$$\left.\begin{array}{ll}
(9.2^e_m): & h_1 \cdots h_m x^e y^2 \approx h_1 \cdots h_m y x^e y, \\
(9.2^e_\infty): & h_1^2 h_2 x^e y^2 \approx h_1^2 h_2 y x^e y, \\
(9.2^e_\dagger): & h^2 x^e y^2 \approx h^2 y x^e y, \\
(9.2^e_\ddagger): & h x^e y^2 h \approx h y x^e y h,
\end{array}\right\} \tag{9.2}$$

$$\left.\begin{array}{ll}
(9.3^e_m): & x^2 y^e t_1 \cdots t_m \approx x y^e x t_1 \cdots t_m, \\
(9.3^e_\infty): & x^2 y^e t_1 t_2^2 \approx x y^e x t_1 t_2^2, \\
(9.3^e_\dagger): & x^2 y^e t^2 \approx x y^e x t^2, \\
(9.3^e_\ddagger): & t x^2 y^e t \approx t x y^e x t,
\end{array}\right\} \tag{9.3}$$

where $m \in \{0, 1, \ldots\}$ and $e \in \{1, 2\}$.

Lemma 9.1.

(i) $\mathbf{S}_3\{(9.2^1_\dagger), (9.3^1_\ddagger)\} = \mathbf{S}_3\{(9.3^1_\dagger), (9.2^1_\ddagger)\}$.
(ii) $\mathbf{S}_3\{(9.2^2_\dagger)\} = \mathbf{S}_3\{(9.2^2_\ddagger)\}$.

Proof. (i) The deduction $(9.3^1_\dagger) \Vdash (9.3^1_\ddagger)$ is easy to establish. Since

$$\big\{\big(9.3^1_\dagger\big), \big(9.2^1_\ddagger\big)\big\} \Vdash x^2yzy \overset{(4.1)}{\approx} x^2yzy^7 \overset{(9.3^1_\dagger)}{\approx} xyzxy^7$$

$$\begin{aligned} &\overset{(4.1)}{\approx} \big(xyzy^6x\big)y \overset{(9.2^1_\ddagger)}{\approx} xzy^7xy \\ &\overset{(4.1)}{\approx} xzyxy \overset{(4.3)}{\approx} xzxy^2 \overset{(9.3^1_\dagger)}{\approx} x^2zy^2 \\ \vdash \big(9.2^1_\dagger\big), \end{aligned}$$

the inclusion $\mathbf{S}_3\{(9.2^1_\dagger), (9.3^1_\ddagger)\} \supseteq \mathbf{S}_3\{(9.3^1_\dagger), (9.2^1_\ddagger)\}$ holds. By symmetry, the inclusion $\mathbf{S}_3\{(9.2^1_\dagger), (9.3^1_\ddagger)\} \subseteq \mathbf{S}_3\{(9.3^1_\dagger), (9.2^1_\ddagger)\}$ also holds.

(ii) The inclusion $\mathbf{S}_3\{(9.2^2_\dagger)\} \subseteq \mathbf{S}_3\{(9.2^2_\ddagger)\}$ is easy to establish. Since

$$\begin{aligned} \big(9.2^2_\ddagger\big) \Vdash h^2x^2y^2 &\overset{(9.4)}{\approx} hxyhxy \overset{(4.1)}{\approx} hx^7y^7hxy \overset{(4.2)}{\approx} hx^2y^2hx^6y^6 \\ &\overset{(9.2^2_\ddagger)}{\approx} hyx^2yhx^6y^6 \overset{(4.2)}{\approx} hyxhx^7y^7 \overset{(4.1)}{\approx} (hyxhx)y \\ &\overset{(4.3)}{\approx} hyhx^2y \overset{(4.3)}{\approx} h^2yx^2y \\ \vdash \big(9.2^2_\dagger\big), \end{aligned}$$

the inclusion $\mathbf{S}_3\{(9.2^2_\dagger)\} \supseteq \mathbf{S}_3\{(9.2^2_\ddagger)\}$ holds. □

Lemma 9.2. *The identities* (4.1) *and* $xyzx \approx xzyx$ *imply the identities* (9.1)*.*

Proof. The identities (4.1) and $xyzx \approx xzyx$ imply the first identity in (9.1) since

$$xhyztx \overset{(4.1)}{\approx} xh\big(x^6yzx^6\big)tx \approx xhx^6zyx^6tx \overset{(4.1)}{\approx} xhzytx.$$

The other deductions can be obtained very similarly. □

Lemma 9.3. *The identities* Ⓢ *imply the identity*

$$(x_1 \cdots x_n)^r \approx x_1^r \cdots x_n^r \tag{9.4}$$

for any $n, r \geqslant 2$*.*

Proof. Let α_n^r denote the identity $(x_1 \cdots x_n)^r \approx x_1^r \cdots x_n^r$. The identities Ⓢ clearly imply the identity α_2^2. Since

$$\begin{aligned} \alpha_2^k \Vdash (x_1x_2)^{k+1} &= (x_1x_2)^kx_1x_2 \overset{\alpha_2^k}{\approx} x_1^kx_2^kx_1x_2 \\ &\overset{(4.2)}{\approx} x_1^kx_2x_1x_2^k \overset{(4.3)}{\approx} x_1^{k+1}x_2^{k+1} \\ \vdash \alpha_2^{k+1}, \end{aligned}$$

the identities Ⓢ imply the identity α_2^r for any $r \geqslant 2$. Denote by φ the substitution $x_k \mapsto x_kx_{k+1}$. Since

$$
\begin{aligned}
\{\alpha_2^r, \alpha_k^r\} \vdash (x_1 \cdots x_{k-1} x_k x_{k+1})^r &= \big((x_1 \cdots x_{k-1} x_k)^r\big)\varphi \\
&\overset{\alpha_k^r}{\approx} \big(x_1^r \cdots x_{k-1}^r x_k^r\big)\varphi \\
&= x_1^r \cdots x_{k-1}^r (x_k x_{k+1})^r \\
&\overset{\alpha_2^r}{\approx} x_1^r \cdots x_{k-1}^r x_k^r x_{k+1}^r \\
\vdash \alpha_{k+1}^r,
\end{aligned}
$$

the identities Ⓢ imply the identity α_n^r for any $n, r \geqslant 2$. $\square$

Let x and y be any letters in a word $\mathbf{w}$. The expression $x \prec_{\mathbf{w}} y$ indicates the condition that x *precedes* y *in* $\mathbf{w}$, that is, within $\mathbf{w}$, the first x occurs before the first y.

Lemma 9.4. *Let $\mathbf{u} \approx \mathbf{v}$ be any balanced identity that is not a permutation identity. Suppose that y and z are simple letters of $\mathbf{u} \approx \mathbf{v}$ such that*

$$\mathbf{u} = \mathbf{u}_1 y \mathbf{u}_2 z \mathbf{u}_3 \quad \text{and} \quad \mathbf{v} = \mathbf{v}_1 z \mathbf{v}_2 y \mathbf{v}_3$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in (\mathcal{X} \backslash \{y, z\})^*$ and all letters of $\mathbf{u}_2$ are non-simple in $\mathbf{u}$. Then

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, \Lambda\}$$

where $\mathbf{u}^\star = \mathbf{u}_1 z \mathbf{u}_2 y \mathbf{u}_3$ and Λ is some subset of $\{(9.1), (9.2), (9.3)\}$.

Proof. Denote by φ_1 the substitution $p \mapsto x^6$ for all $p \in \mathcal{X} \backslash \{y, z\}$. Since

$$\mathbf{u} \approx \mathbf{v} \Vdash xyzx \overset{(4.1)}{\approx} x(\mathbf{u}\varphi_1)x \approx x(\mathbf{v}\varphi_1)x \overset{(4.1)}{\approx} xzyx \vdash xyzx \approx xzyx,$$

it follows from Lemma 9.2 that

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}, (9.1)\}. \tag{9.5}$$

If $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_3) \neq \emptyset$, say $x \in \mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_3)$, then since

$$\mathbf{u} \overset{(4.1)}{\approx} \mathbf{u}_1 x^6 y \mathbf{u}_2 z x^6 \mathbf{u}_3 \overset{(9.1)}{\approx} \mathbf{u}_1 x^6 z \mathbf{u}_2 y x^6 \mathbf{u}_3 \overset{(4.1)}{\approx} \mathbf{u}^\star,$$

it follows from (9.5) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.1)\}$. Therefore it suffices to assume that $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_3) = \emptyset$. There are four cases.

Case 1. $\mathbf{u}_1$ and $\mathbf{u}_3$ each contains some non-simple letter of $\mathbf{u}$.

1.1. $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_2) \neq \emptyset$ and $\mathsf{C}(\mathbf{u}_2) \cap \mathsf{C}(\mathbf{u}_3) \neq \emptyset$. By assumption, $\mathbf{u}_1 = \mathbf{u}_1' h \mathbf{u}_1''$ and $\mathbf{u}_3 = \mathbf{u}_3' t \mathbf{u}_3''$ for some $\mathbf{u}_1', \mathbf{u}_1'', \mathbf{u}_3', \mathbf{u}_3'' \in \mathcal{X}^*$ and $h, t \in \mathsf{C}(\mathbf{u}_2)$. Denote by ϵ_7 the substitution $p \mapsto p^7$ for all $p \in \mathcal{X}$. Since all letters in $\mathbf{w} = h^6 \mathbf{u}_2 t^6$ are non-simple in $\mathbf{u}$,

$$
\begin{aligned}
\mathbf{u} &\overset{(4.1)}{\approx} \mathbf{u}_1' h \mathbf{u}_1'' y \mathbf{w} z \mathbf{u}_3' t \mathbf{u}_3'' \\
&\overset{(4.1)}{\approx} \mathbf{u}_1' h \mathbf{u}_1'' y (\mathbf{w}\epsilon_7) z \mathbf{u}_3' t \mathbf{u}_3''
\end{aligned}
$$

$$\overset{(9.4)}{\approx} \mathbf{u}_1' h \mathbf{u}_1'' y \mathbf{w}^7 z \mathbf{u}_3' t \mathbf{u}_3''$$
$$\overset{(9.1)}{\approx} \mathbf{u}_1 \mathbf{w} y \mathbf{w}^5 z \mathbf{w} \mathbf{u}_3.$$

Therefore the deduction (9.1) $\Vdash \mathbf{u} \approx \mathbf{u}_1 \mathbf{w} y \mathbf{w}^5 z \mathbf{w} \mathbf{u}_3$ holds. Similarly, the deduction (9.1) $\Vdash \mathbf{u}^\star \approx \mathbf{u}_1 \mathbf{w} z \mathbf{w}^5 y \mathbf{w} \mathbf{u}_3$ also holds. Since

$$(9.1) \Vdash \mathbf{u} \overset{(9.1)}{\approx} \mathbf{u}_1 \mathbf{w} y \mathbf{w}^5 z \mathbf{w} \mathbf{u}_3 \overset{(9.1)}{\approx} \mathbf{u}_1 \mathbf{w} z \mathbf{w}^5 y \mathbf{w} \mathbf{u}_3 \overset{(9.1)}{\approx} \mathbf{u}^\star,$$

it follows from (9.5) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.1)\}$.

1.2. $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_2) = \emptyset$. Denote by φ_2 the substitution

$$p \mapsto \begin{cases} x^6 & \text{if } p \in \mathsf{C}(\mathbf{u}_1), \\ x^6 y & \text{if } p = y, \\ z^6 & \text{if } p \in \mathsf{C}(\mathbf{u}_2 z \mathbf{u}_3). \end{cases}$$

Then $x^2(\mathbf{u}\varphi_2)z^2 \overset{(4.1)}{\approx} x^2 y z^2$ and

$$\begin{aligned} x^2(\mathbf{v}\varphi_2)z^2 &= x^2(\mathbf{v}_1\varphi_2)z^6(\mathbf{v}_2\varphi_2)x^6 y(\mathbf{v}_3\varphi_2)z^2 \\ &\overset{(9.1)}{\approx} x^2(\mathbf{v}_1\varphi_2)z^6(\mathbf{v}_2\varphi_2)(\mathbf{v}_3\varphi_2)x^6 y z^2 \\ &\overset{(4.2)}{\approx} x^7(\mathbf{v}_1\varphi_2)z^7(\mathbf{v}_2\varphi_2)(\mathbf{v}_3\varphi_2)xyz \\ &\overset{(4.1)}{\approx} xzxyz \\ &\overset{(4.3)}{\approx} x^2 zyz, \end{aligned}$$

where the second last deduction holds since $\mathbf{v}_1\varphi_2$, $\mathbf{v}_2\varphi_2$, and $\mathbf{v}_3\varphi_2$ are words over $\{x^6, z^6\}$. Hence $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}, (9.1), (9.2^1_\dagger)\}$ by (9.5). Since

$$\begin{aligned} \mathbf{u} &\overset{(4.1)}{\approx} \mathbf{u}_1' h^7 \mathbf{u}_1'' y \mathbf{u}_2 z \mathbf{u}_3' t^7 \mathbf{u}_3'' \\ &\overset{(9.2^1_\dagger)}{\approx} \mathbf{u}_1' h^7 t \mathbf{u}_1'' y \mathbf{u}_2 z \mathbf{u}_3' t^6 \mathbf{u}_3'' \\ &\overset{(9.1)}{\approx} \mathbf{u}_1' h^7 t \mathbf{u}_1'' z \mathbf{u}_2 y \mathbf{u}_3' t^6 \mathbf{u}_3'' \\ &\overset{(9.2^1_\dagger)}{\approx} \mathbf{u}_1' h^7 \mathbf{u}_1'' z \mathbf{u}_2 y \mathbf{u}_3' t^7 \mathbf{u}_3'' \\ &\overset{(4.1)}{\approx} \mathbf{u}^\star, \end{aligned}$$

it follows that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.1), (9.2^1_\dagger)\}$.

1.3. $\mathsf{C}(\mathbf{u}_2) \cap \mathsf{C}(\mathbf{u}_3) = \emptyset$. Then $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.1), (9.3^1_\dagger)\}$ by an argument symmetrical to Subcase 1.2.

Case 2. Every letter in $\mathbf{u}_1$ is simple in $\mathbf{u}$ and some letter in $\mathbf{u}_3$ is non-simple in $\mathbf{u}$. Suppose that $\mathbf{u}_1 = x_1 \cdots x_\ell$ for some $\ell \geqslant 0$. Let m be the largest such that $x_1 \cdots x_m$ is a prefix of $\mathbf{v}$. Since $z \notin \{x_1, \ldots, x_\ell\}$, the word $x_1 \cdots x_m$ is necessarily a prefix of $\mathbf{v}_1$. Let h be the letter in $\mathbf{u}$ that immediately follows x_m. (Then

$$h = \begin{cases} x_{m+1} & \text{if } m < \ell, \\ y & \text{if } m = \ell, \end{cases}$$

and h is simple in $\mathbf{u} \approx \mathbf{v}$. It follows that h is not the letter in $\mathbf{v}$ that immediately follows x_m.) Denote by φ_3 the substitution $p \mapsto z^6$ for all $p \in \mathcal{X} \backslash \{x_1, \ldots, x_m, h\}$. Since

$$\begin{aligned} \mathbf{u} \approx \mathbf{v} \Vdash x_1 \cdots x_m h z^2 &\overset{(4.1)}{\approx} (\mathbf{u}\varphi_3) z^2 \approx (\mathbf{v}\varphi_3) z^2 \\ &\overset{(4.1)}{\approx} x_1 \cdots x_m z^6 h z^2 \\ &\overset{(4.2)}{\approx} x_1 \cdots x_m z^7 h z \\ &\overset{(4.1)}{\approx} x_1 \cdots x_m z h z \\ \vdash \big(9.2_m^1\big), \end{aligned}$$

it follows from (9.5) that $\mathbf{S}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}\{\mathbf{u} \approx \mathbf{v}, (9.1), (9.2_m^1)\}$. By assumption, the word $\mathbf{u}_3$ contains some non-simple letter, say t. Since

$$\begin{aligned} \mathbf{u} &\overset{(4.1)}{\approx} x_1 \cdots x_m \cdots y \mathbf{u}_2 z \cdots t^7 \cdots \\ &\overset{(9.2_m^1)}{\approx} x_1 \cdots x_m t \cdots y \mathbf{u}_2 z \cdots t^6 \cdots \\ &\overset{(9.1)}{\approx} x_1 \cdots x_m t \cdots z \mathbf{u}_2 y \cdots t^6 \cdots \\ &\overset{(9.2_m^1)}{\approx} x_1 \cdots x_m \cdots z \mathbf{u}_2 y \cdots t^7 \cdots \\ &\overset{(4.1)}{\approx} \mathbf{u}^\star, \end{aligned}$$

it follows that $\mathbf{S}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}\{\mathbf{u}^\star \approx \mathbf{v}, (9.1), (9.2_m^1)\}$.

Case 3. Some letter in $\mathbf{u}_1$ is non-simple in $\mathbf{u}$ and every letter in $\mathbf{u}_3$ is simple in $\mathbf{u}$. This is symmetrical to Case 2. Hence $\mathbf{S}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}\{\mathbf{u}^\star \approx \mathbf{v}, (9.1), (9.3_m^1)\}$ for some $m \geqslant 0$.

Case 4. Every letter in $\mathbf{u}_1$ and $\mathbf{u}_3$ is simple in $\mathbf{u}$. Suppose that $\mathbf{u}_1 = x_1 \cdots x_\ell$ and $\mathbf{u}_3 = z_r \cdots z_1$ for some $\ell, r \geqslant 0$. Let m and n be the largest such that $x_1 \cdots x_m$ is a prefix of $\mathbf{v}$ and $z_n \cdots z_1$ is a suffix of $\mathbf{v}$. It follows from arguments in Cases 2 and 3 that $\mathbf{S}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}\{\mathbf{u} \approx \mathbf{v}, (9.1), (9.2_m^1), (9.3_n^1)\}$. By assumption, all letters in $\mathbf{u}_2$ are non-simple in $\mathbf{u}$; let t be one of these non-simple letters. Since

$$\begin{aligned} \mathbf{u} &\overset{(4.1)}{\approx} x_1 \cdots x_m \cdots y \cdots t^7 \cdots z \cdots z_n \cdots z_1 \\ &\overset{(9.2_m^1)}{\approx} x_1 \cdots x_m t \cdots y \cdots t^6 \cdots z \cdots z_n \cdots z_1 \\ &\overset{(9.3_n^1)}{\approx} x_1 \cdots x_m t \cdots y \cdots t^5 \cdots z \cdots t z_n \cdots z_1 \\ &\overset{(9.1)}{\approx} x_1 \cdots x_m t \cdots z \cdots t^5 \cdots y \cdots t z_n \cdots z_1 \\ &\overset{(9.2_m^1)}{\approx} x_1 \cdots x_m \cdots z \cdots t^6 \cdots y \cdots t z_n \cdots z_1 \\ &\overset{(9.3_n^1)}{\approx} x_1 \cdots x_m \cdots z \cdots t^7 \cdots y \cdots z_n \cdots z_1 \\ &\overset{(4.1)}{\approx} \mathbf{u}^\star, \end{aligned}$$

it follows that $\mathbf{S}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}\{\mathbf{u}^\star \approx \mathbf{v}, (9.1), (9.2_m^1), (9.3_n^1)\}$. $\square$

Lemma 9.5. *Let* $\mathbf{u} \approx \mathbf{v}$ *be any balanced identity that satisfies the property*

(P) $p \prec_{\mathbf{u}} q$ *if and only if* $p \prec_{\mathbf{v}} q$

for any of its simple letters p *and* q. *Suppose that* y *is a simple letter and* z *is a non-simple letter of* $\mathbf{u} \approx \mathbf{v}$ *such that* $y \prec_{\mathbf{u}} z$ *and* $z \prec_{\mathbf{v}} y$, *say*

$$\mathbf{u} = \mathbf{u}_1 y \mathbf{u}_2 z \mathbf{u}_3 \quad \textit{and} \quad \mathbf{v} = \mathbf{v}_1 z \mathbf{v}_2 y \mathbf{v}_3$$

for some $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in (\mathcal{X} \backslash \{y\})^*$ *with* $z \in \mathsf{C}(\mathbf{u}_3\mathbf{v}_2\mathbf{v}_3) \backslash \mathsf{C}(\mathbf{u}_1\mathbf{u}_2\mathbf{v}_1)$. *Then*

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, \Lambda\}$$

where $\mathbf{u}^\star = \mathbf{u}_1 z y \mathbf{u}_2 \mathbf{u}_3$ *and* Λ *is some subset of* (9.2).

Proof. Denote by φ_1 the substitution

$$p \mapsto \begin{cases} x^6 & \text{if } p \in \mathcal{X} \backslash \{y, z\}, \\ yz^2 & \text{if } p = y, \\ z^6 & \text{if } p = z. \end{cases}$$

Since

$$\mathbf{u} \approx \mathbf{v} \Vdash xyz^2x \overset{(4.1)}{\approx} x(\mathbf{u}\varphi_1)x \approx x(\mathbf{v}\varphi_1)x \overset{(4.1)}{\approx} xz^6yz^2x \overset{(4.2)}{\approx} xz^7yzx \overset{(4.1)}{\approx} xzyzx$$
$$\vdash \left(9.2^1_\ddagger\right),$$

it follows that

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}, \left(9.2^1_\ddagger\right)\}. \tag{9.6}$$

By assumption, $\mathbf{u}_3 = \mathbf{u}_3' z \mathbf{u}_3''$ for some $\mathbf{u}_3', \mathbf{u}_3'' \in \mathcal{X}^*$. If $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_3) \neq \emptyset$, say $x \in \mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_3) \neq \emptyset$, then since

$$\begin{aligned} \mathbf{u} &\overset{(4.1)}{\approx} \mathbf{u}_1\left(x^6 y \mathbf{u}_2 z^7 x^6\right)\mathbf{u}_3' z \mathbf{u}_3'' \\ &\overset{(9.2^1_\ddagger)}{\approx} \mathbf{u}_1 x^6\left(zy\mathbf{u}_2 z^6 x^6 \mathbf{u}_3' z\right)\mathbf{u}_3'' \\ &\overset{(4.1)}{\approx} \mathbf{u}^\star, \end{aligned}$$

it follows from (9.6) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.2^1_\ddagger)\}$. Therefore it suffices to assume that $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_3) = \emptyset$. There are two cases.

Case 1. Every letter in $\mathbf{u}_1$ is simple in $\mathbf{u}$. Then $\mathbf{u}_1 = x_1 \cdots x_\ell$ for some $\ell \geqslant 0$. The letters $x_1, \ldots, x_\ell, y$ are simple in $\mathbf{v}$ and so by (P), they occur in $\mathbf{v}$ in the same order relative to each other. Therefore $\mathbf{v}_1 z \mathbf{v}_2 = \mathbf{w}_0 x_1 \mathbf{w}_1 \cdots x_\ell \mathbf{w}_\ell$ for some $\mathbf{w}_0, \ldots, \mathbf{w}_\ell \in \mathcal{X}^*$ such that the letters of $\mathbf{w}_0, \ldots, \mathbf{w}_\ell$ are all non-simple in $\mathbf{v}$. (Note that z is a letter in some $\mathbf{w}_i$ so that the words $\mathbf{w}_0, \ldots, \mathbf{w}_\ell$ cannot be all empty.) Denote by φ_2 the substitution $p \mapsto z^6$ for all $p \in \mathcal{X} \backslash \{x_1, \ldots, x_\ell, y\}$. Since

$$(\mathbf{u}\varphi_2)z^2 \overset{(4.1)}{\approx} x_1\cdots x_\ell yz^2,$$
$$(\mathbf{v}\varphi_2)z^2 \overset{(4.1)}{\approx} (\mathbf{w}_0\varphi_2)x_1(\mathbf{w}_1\varphi_2)\cdots x_\ell(\mathbf{w}_\ell\varphi_2)yz^2,$$

the identities in $\{\mathbf{u}\approx\mathbf{v}, (4.1)\}$ imply the identity

$$x_1\cdots x_\ell yz^2 \approx (\mathbf{w}_0\varphi_2)x_1(\mathbf{w}_1\varphi_2)\cdots x_\ell(\mathbf{w}_\ell\varphi_2)yz^2. \tag{9.7}$$

As observed earlier, some $\mathbf{w}_i$ is nonempty so that $\mathbf{w}_i\varphi_2 \overset{(4.1)}{\approx} z^6$ for some i. Therefore

$$\begin{aligned}(9.7) \Vdash x_1\cdots x_\ell zyz &\overset{(4.1)}{\approx} x_1\cdots x_\ell z^7yz\\ &\overset{(4.2)}{\approx} x_1\cdots x_\ell\big(z^6y\big)z^2\\ &\overset{(9.7)}{\approx} (\mathbf{w}_0\varphi_2)x_1(\mathbf{w}_1\varphi_2)\cdots x_\ell(\mathbf{w}_\ell\varphi_2)\big(z^6y\big)z^2\\ &\overset{(4.1)}{\approx} (\mathbf{w}_0\varphi_2)x_1(\mathbf{w}_1\varphi_2)\cdots x_\ell(\mathbf{w}_\ell\varphi_2)yz^2\\ &\overset{(9.7)}{\approx} x_1\cdots x_\ell yz^2\\ \vdash \big(9.2^1_\ell\big),&\end{aligned}$$

whence $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}, (9.2^1_\ell), (9.2^1_\ddagger)\}$ by (9.6). Since

$$\begin{aligned}\mathbf{u} &\overset{(4.1)}{\approx} \big(x_1\cdots x_\ell y\mathbf{u}_2z^7\big)\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(9.2^1_\ell)}{\approx} x_1\cdots x_\ell\big(zy\mathbf{u}_2z^6\mathbf{u}_3'z\big)\mathbf{u}_3''\\ &\overset{(4.1)}{\approx} \mathbf{u}^\star,\end{aligned}$$

it follows that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star\approx\mathbf{v}, (9.2^1_\ell), (9.2^1_\ddagger)\}$.

Case 2. Some letter in $\mathbf{u}_1$ is non-simple in $\mathbf{u}$.

2.1. $\mathsf{C}(\mathbf{u}_1)\cap\mathsf{C}(\mathbf{u}_2)\neq\emptyset$. Then $\mathbf{u}_1 = \mathbf{u}_1'h\mathbf{u}_1''$ and $\mathbf{u}_2 = \mathbf{u}_2'h\mathbf{u}_2''$ for some $h\in\mathcal{X}$ and $\mathbf{u}_1', \mathbf{u}_1'', \mathbf{u}_2', \mathbf{u}_2''\in\mathcal{X}^*$. Denote by ϵ_7 the substitution $p\mapsto p^7$ for all $p\in\mathcal{X}$. Since the letters in $\mathbf{u}_2$ are all non-simple in $\mathbf{u}$ by assumption,

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_1'h\mathbf{u}_1''y\mathbf{u}_2'h\mathbf{u}_2''z\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(4.1)}{\approx} \mathbf{u}_1'h\mathbf{u}_1''y\mathbf{u}_2'h\big(\mathbf{u}_2''z\big)\epsilon_7\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(9.4)}{\approx} \mathbf{u}_1'\big(h\mathbf{u}_1''y\mathbf{u}_2'h\big(\mathbf{u}_2''z\big)^7\big)\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(4.3)}{\approx} \mathbf{u}_1'h\mathbf{u}_1''y\mathbf{u}_2'\big(\mathbf{u}_2''z\big)h\big(\mathbf{u}_2''z\big)^6\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(9.4)}{\approx} \mathbf{u}_1'h\mathbf{u}_1''y\mathbf{u}_2'\big(\mathbf{u}_2''z\big)h\big(\mathbf{u}_2''\big)^6z^6\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(4.1)}{\approx} \mathbf{u}_1'h\mathbf{u}_1''\big(h^6y\mathbf{u}_2'\mathbf{u}_2''z^7h\big)\big(\mathbf{u}_2''\big)^6\mathbf{u}_3'z\mathbf{u}_3''\end{aligned}$$

$$
\begin{aligned}
&\overset{(9.2^1_\ddagger)}{\approx} \mathbf{u}_1' h \mathbf{u}_1'' h^6 \big(zy\mathbf{u}_2'\mathbf{u}_2'' z^6 h\big(\mathbf{u}_2''\big)^6 \mathbf{u}_3' z\big)\mathbf{u}_3'' \\
&\overset{(4.1)}{\approx} \mathbf{u}_1' \big(h\mathbf{u}_1'' zy \mathbf{u}_2'\mathbf{u}_2'' h\big(\mathbf{u}_2''\big)^6\big)\mathbf{u}_3' z \mathbf{u}_3'' \\
&\overset{(4.3)}{\approx} \mathbf{u}_1' h\mathbf{u}_1'' zy\mathbf{u}_2' h\big(\mathbf{u}_2''\big)^7 \mathbf{u}_3' z\mathbf{u}_3'' \\
&\overset{(9.4)}{\approx} \mathbf{u}_1' h\mathbf{u}_1'' zy\mathbf{u}_2' h\big(\mathbf{u}_2''\big)\epsilon_7 \mathbf{u}_3' z\mathbf{u}_3'' \\
&\overset{(4.1)}{\approx} \mathbf{u}_1' h\mathbf{u}_1'' zy\mathbf{u}_2' h\mathbf{u}_2''\mathbf{u}_3' z\mathbf{u}_3'' \\
&= \mathbf{u}^\star.
\end{aligned}
$$

Therefore it follows from (9.6) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.2^1_\ddagger)\}$.

2.2. $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_2) = \emptyset$. Denote by φ_3 the substitution

$$
p \mapsto \begin{cases} x^6 & \text{if } p \in \mathsf{C}(\mathbf{u}_1), \\ x^6 y & \text{if } p = y, \\ z^6 & \text{if } p \in \mathsf{C}(\mathbf{u}_2 z \mathbf{u}_3). \end{cases}
$$

Then $x^2(\mathbf{u}\varphi_3)z^2 \overset{(4.1)}{\approx} x^2yz^2$ and

$$
\begin{aligned}
x^2(\mathbf{v}\varphi_3)z^2 &= x^2(\mathbf{v}_1\varphi_3)z^6(\mathbf{v}_2\varphi_3)x^6y(\mathbf{v}_3\varphi_3)z^2 \\
&\overset{(4.2)}{\approx} x^7(\mathbf{v}_1\varphi_3)z^7(\mathbf{v}_2\varphi_3)xy(\mathbf{v}_3\varphi_3)z \\
&\overset{(4.1)}{\approx} x(\mathbf{v}_1\varphi_3)z(\mathbf{v}_2\varphi_3)xy(\mathbf{v}_3\varphi_3)z,
\end{aligned}
$$

where $\mathbf{v}_1\varphi_3$, $\mathbf{v}_2\varphi_3$, and $\mathbf{v}_3\varphi_3$ are possibly empty words over $\{x^6, z^6\}$. By applying the identities (4.1) to the word $x(\mathbf{v}_1\varphi_3)z(\mathbf{v}_2\varphi_3)xy(\mathbf{v}_3\varphi_3)z$,

- the factors $\mathbf{v}_1\varphi_3$ and $\mathbf{v}_2\varphi_3$ can be eliminated,
- all multiples of z^6 can be eliminated from $\mathbf{v}_3\varphi_3$, and
- if x^6 is some factor of $\mathbf{v}_3\varphi_3$, then all except one multiple of x^6 can be eliminated.

Therefore $x(\mathbf{v}_1\varphi_3)z(\mathbf{v}_2\varphi_3)xy(\mathbf{v}_3\varphi_3)z \overset{(4.1)}{\approx} xzxyx^rz$ for some $r \in \{0, 6\}$, whence the identities in Ⓢ $\cup$ $\{\mathbf{u} \approx \mathbf{v}\}$ imply the identity

$$
x^2yz^2 \approx xzxyx^rz. \tag{9.8}
$$

If $r = 6$ in (9.8), then

$$
\begin{aligned}
\big\{\big(9.2^1_\ddagger\big), (9.8)\big\} \Vdash x^2yz^2 &\overset{(9.8)}{\approx} xzxyx^6z \overset{(4.2)}{\approx} xzyx^7z \\
&\overset{(4.1)}{\approx} \big(xzyz^6x\big)z \overset{(9.2^1_\ddagger)}{\approx} xyz^7xz \\
&\overset{(4.1)}{\approx} xyzxz \overset{(4.3)}{\approx} xyxz^2 \\
&\vdash \big(9.3^1_\dagger\big),
\end{aligned}
$$

whence the deduction $\{(9.2^1_\ddagger), \mathbf{u} \approx \mathbf{v}\} \Vdash (9.2^1_\dagger)$ holds by Lemma 9.1(i); if $r = 0$ in (9.8), then this deduction also holds since

$$\begin{aligned}\left\{\left(9.2^1_\ddagger\right), \mathbf{u} \approx \mathbf{v}\right\} &\Vdash (9.8)\\ &\Vdash x^2yz^2 \overset{(9.8)}{\approx} xzxyz \overset{(4.3)}{\approx} x^2zyz\\ &\vdash \left(9.2^1_\dagger\right).\end{aligned}$$

Therefore it follows from (9.6) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}, (9.2^1_\dagger), (9.2^1_\ddagger)\}$ independent of the value of r. Since

$$\begin{aligned}\mathbf{u} &\overset{(4.1)}{\approx} \mathbf{u}_1'\left(h^7\mathbf{u}_1''y\mathbf{u}_2z^7\right)\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(9.2^1_\dagger)}{\approx} \mathbf{u}_1'h^7\left(z\mathbf{u}_1''y\mathbf{u}_2z^6\mathbf{u}_3'z\right)\mathbf{u}_3''\\ &\overset{(4.1)}{\approx} \mathbf{u}_1'\left(h^7z\mathbf{u}_1''z^6\right)y\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(9.2^1_\dagger)}{\approx} \mathbf{u}_1'h^7\mathbf{u}_1''z^7y\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3''\\ &\overset{(4.1)}{\approx} \mathbf{u}^\star,\end{aligned}$$

it follows that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.2^1_\dagger), (9.2^1_\ddagger)\}$. $\square$

Lemma 9.6. *Let $\mathbf{u} \approx \mathbf{v}$ be any balanced identity that satisfies the property*

(Q) *$p \prec_\mathbf{u} q$ if and only if $p \prec_\mathbf{v} q$*

for all pairs $(p; q)$ such that p is a simple letter and q is a non-simple letter of $\mathbf{u} \approx \mathbf{v}$. Suppose that y and z are non-simple letters of $\mathbf{u} \approx \mathbf{v}$ such that $y \prec_\mathbf{u} z$ and $z \prec_\mathbf{v} y$, say

$$\mathbf{u} = \mathbf{u}_1y\mathbf{u}_2z\mathbf{u}_3 \quad \textit{and} \quad \mathbf{v} = \mathbf{v}_1z\mathbf{v}_2y\mathbf{v}_3$$

for some $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{X}^*$ with $y \in \mathsf{C}(\mathbf{u}_2\mathbf{u}_3)\backslash\mathsf{C}(\mathbf{u}_1)$, $z \in \mathsf{C}(\mathbf{u}_3)\backslash\mathsf{C}(\mathbf{u}_1\mathbf{u}_2)$, $y \in \mathsf{C}(\mathbf{v}_3)\backslash\mathsf{C}(\mathbf{v}_1\mathbf{v}_2)$, and $z \in \mathsf{C}(\mathbf{v}_2\mathbf{v}_3)\backslash\mathsf{C}(\mathbf{v}_1)$. Then

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\left\{\mathbf{u}^\star \approx \mathbf{v}, \Lambda\right\}$$

where $\mathbf{u}^\star = \mathbf{u}_1zy\mathbf{u}_2\mathbf{u}_3$ and Λ is some subset of (9.2).

Proof. Suppose that $x \in \mathsf{C}(\mathbf{u}_2)$ is simple in $\mathbf{u}$ so that $x \prec_\mathbf{u} z$. Then $x \prec_\mathbf{v} z$ by (Q), whence $x \in \mathsf{C}(\mathbf{v}_1)$. It follows that $x \prec_\mathbf{v} y$, which is impossible since $x \not\prec_\mathbf{u} y$. Consequently, every letter in $\mathbf{u}_2$ is non-simple in $\mathbf{u}$.

Denote by φ_1 the substitution

$$p \mapsto \begin{cases} x^6 & \text{if } p \in \mathcal{X}\backslash\{y, z\},\\ y^6 & \text{if } p = y,\\ z^6 & \text{if } p = z.\end{cases}$$

Then the deduction $\mathbf{u} \approx \mathbf{v} \Vdash (9.2^2_{\ddagger})$ holds since

$$x(\mathbf{u}\varphi_1)y^2z^2x \overset{(4.1)}{\approx} xy^6z^6y^2z^2x \overset{(4.2)}{\approx} xy^7z^7yzx \overset{(4.1)}{\approx} xyzyzx \overset{(4.3)}{\approx} xy^2z^2x,$$
$$x(\mathbf{v}\varphi_1)y^2z^2x \overset{(4.1)}{\approx} xz^6y^2z^2x \overset{(4.2)}{\approx} xz^7y^2zx \overset{(4.1)}{\approx} xzy^2zx.$$

Therefore it follows from Lemma 9.1(ii) that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}, (9.2^2_{\dagger})\}$. By assumption, $\mathbf{u}_3 = \mathbf{u}'_3 z \mathbf{u}''_3$ for some $\mathbf{u}'_3, \mathbf{u}''_3 \in \mathcal{X}^*$. Denote by ϵ_7 the substitution $p \mapsto p^7$ for all $p \in \mathcal{X}$. Since every letter in $\mathbf{u}_2$ is non-simple in $\mathbf{u}$,

$$\begin{aligned}
\mathbf{u} &\overset{(4.1)}{\approx} \mathbf{u}_1 y^7(\mathbf{u}_2\epsilon_7)z^7\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(9.4)}{\approx} \mathbf{u}_1 y^7\mathbf{u}_2^7 z^7\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(4.3)}{\approx} \mathbf{u}_1 y^7\mathbf{u}_2^6 z\mathbf{u}_2 z^6\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(4.2)}{\approx} \mathbf{u}_1\big(y^7\mathbf{u}_2^2 z^2\big)\mathbf{u}_2^5 z^5\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(9.2^2_{\dagger})}{\approx} \mathbf{u}_1 y^7\big(z\mathbf{u}_2^2 z\mathbf{u}_2^5 z^5\mathbf{u}'_3 z\big)\mathbf{u}''_3 \\
&\overset{(4.2)}{\approx} \mathbf{u}_1 y^7 z^7\mathbf{u}_2^7\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(9.4)}{\approx} \mathbf{u}_1 y^7 z^7(\mathbf{u}_2\epsilon_7)\mathbf{u}_3 \\
&\overset{(4.1)}{\approx} \mathbf{u}_1 yz\mathbf{u}_2\mathbf{u}_3.
\end{aligned}$$

Hence

$$\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\big\{\hat{\mathbf{u}} \approx \mathbf{v}, \big(9.2^2_{\dagger}\big)\big\} \quad \text{where } \hat{\mathbf{u}} = \mathbf{u}_1 yz\mathbf{u}_2\mathbf{u}_3. \tag{9.9}$$

There are three cases.

Case 1. $\mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_2\mathbf{u}_3) \neq \emptyset$, say $x \in \mathsf{C}(\mathbf{u}_1) \cap \mathsf{C}(\mathbf{u}_2\mathbf{u}_3)$. Then

$$\left.\begin{aligned}
\hat{\mathbf{u}} &\overset{(4.1)}{\approx} \mathbf{u}_1 x^6 y^7 z^7\mathbf{u}_2\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(4.3)}{\approx} \mathbf{u}_1(x^6y^6z^2)yz^5\mathbf{u}_2\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(9.2^2_{\dagger})}{\approx} \mathbf{u}_1 x^6(zy^6zyz^5\mathbf{u}_2\mathbf{u}'_3 z)\mathbf{u}''_3 \\
&\overset{(4.2)}{\approx} \mathbf{u}_1 x^6 z^7 y^7\mathbf{u}_2\mathbf{u}'_3 z\mathbf{u}''_3 \\
&\overset{(4.1)}{\approx} \mathbf{u}^\star
\end{aligned}\right\} \tag{9.10}$$

so that $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.2^2_{\dagger})\}$ by (9.9).

Case 2. $t = \mathsf{t}(\mathbf{u}_1)$ is non-simple in $\mathbf{u}$. Then $\hat{\mathbf{u}} \overset{(4.1)}{\approx} \mathbf{u}_1 t^6 y^7 z^7\mathbf{u}_2\mathbf{u}'_3 z\mathbf{u}''_3$. Repeat the deductions in (9.10), with $x = t$, to obtain $\mathbf{S}_3\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star \approx \mathbf{v}, (9.2^2_{\dagger})\}$.

Case 3. $\mathsf{C}(\mathbf{u}_1)\cap\mathsf{C}(\mathbf{u}_2\mathbf{u}_3)=\emptyset$ and $t=\mathsf{t}(\mathbf{u}_1)$ is simple in $\mathbf{u}$. Then

(i) $t\prec_{\mathbf{u}} y$, $t\prec_{\mathbf{u}} z$, and
(ii) $x\prec_{\mathbf{u}} t$ if and only if $x\in\mathsf{C}(\mathbf{u}_1)\backslash\{t\}$.

It follows from (i) and (Q) that $t\prec_{\mathbf{v}} y$ and $t\prec_{\mathbf{v}} z$. Hence $t\in\mathsf{C}(\mathbf{v}_1)$ and $\mathbf{v}_1=\mathbf{v}_1't\mathbf{v}_1''$ for some $\mathbf{v}_1',\mathbf{v}_1''\in\mathcal{X}^*$. Further, it follows from (i), (ii), and (Q) that $\mathsf{C}(\mathbf{u}_1)\backslash\{t\}=\mathsf{C}(\mathbf{v}_1')$ and $\mathsf{C}(\mathbf{v}_1')\cap\mathsf{C}(\mathbf{v}_1''z\mathbf{v}_2y\mathbf{v}_3)=\emptyset$. Denote by φ_2 the substitution

$$p\mapsto\begin{cases}x^6 & \text{if } p\in\mathsf{C}(\mathbf{u}_1)\backslash\{t\},\\ y^6 & \text{if } p=y,\\ z^6 & \text{if } p\in\mathsf{C}(z\mathbf{u}_2\mathbf{u}_3)\backslash\{y\}.\end{cases}$$

Then the deduction $\hat{\mathbf{u}}\approx\mathbf{v}\Vdash(9.2^2_\infty)$ holds since

$$x^2(\hat{\mathbf{u}}\varphi_2)y^2z^2\overset{(4.1)}{\approx}x^2ty^6z^6y^2z^2\overset{(4.2)}{\approx}x^2ty^7z^7yz\overset{(4.1)}{\approx}x^2tyzyz\overset{(4.3)}{\approx}x^2ty^2z^2,$$
$$x^2(\mathbf{v}\varphi_2)y^2z^2\overset{(4.1)}{\approx}x^2tz^6y^2z^2\overset{(4.2)}{\approx}x^2tz^7y^2z\overset{(4.1)}{\approx}x^2tzy^2z.$$

Therefore it follows from (9.9) that

$$\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\bigl\{\hat{\mathbf{u}}\approx\mathbf{v},\,\bigl(9.2^2_\infty\bigr),\,\bigl(9.2^2_\dagger\bigr)\bigr\}. \tag{9.11}$$

If $\mathbf{u}_1$ contains some letter x that is non-simple in $\hat{\mathbf{u}}$, say $\mathbf{u}_1=\mathbf{u}_1'x\mathbf{u}_1''$ for some $\mathbf{u}_1',\mathbf{u}_1''\in\mathcal{X}^*$ (with $t=\mathsf{t}(\mathbf{u}_1''))$, then since

$$\begin{aligned}\hat{\mathbf{u}}&\overset{(4.1)}{\approx}\mathbf{u}_1'x^7\mathbf{u}_1''y^7z^7\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3''\\&\overset{(4.3)}{\approx}\mathbf{u}_1'\bigl(x^7\mathbf{u}_1''y^6z^2\bigr)yz^5\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3''\\&\overset{(9.2^2_\infty)}{\approx}\mathbf{u}_1'x^7\mathbf{u}_1''\bigl(zy^6zyz^5\mathbf{u}_2\mathbf{u}_3'z\bigr)\mathbf{u}_3''\\&\overset{(4.2)}{\approx}\mathbf{u}_1'x^7\mathbf{u}_1''z^7y^7\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3''\\&\overset{(4.1)}{\approx}\mathbf{u}^\star,\end{aligned}$$

it follows that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\}=\mathbf{S}_3\{\mathbf{u}^\star\approx\mathbf{v},(9.2^2_\infty),(9.2^2_\dagger)\}$.

It remains to assume that every letter of $\mathbf{u}_1$ is simple in $\hat{\mathbf{u}}$, say $\mathbf{u}_1=x_1\cdots x_n$ for some $n\geqslant 0$ (where $x_n=t$). As observed earlier, $\mathsf{C}(\mathbf{v}')=\mathsf{C}(\mathbf{u}_1)\backslash\{t\}$. Therefore by (Q), the prefix $\mathbf{v}_1't$ of $\mathbf{v}$ must be $x_1\cdots x_n$. Denote by φ_3 the substitution

$$p\mapsto\begin{cases}y^6 & \text{if } p=y,\\ z^6 & \text{if } p\in\mathsf{C}(z\mathbf{u}_2\mathbf{u}_3)\backslash\{y\}.\end{cases}$$

Then the deduction $\hat{\mathbf{u}}\approx\mathbf{v}\Vdash(9.2^2_n)$ holds since

$$
\begin{aligned}
(\hat{\mathbf{u}}\varphi_3)y^2z^2 &\overset{(4.1)}{\approx} x_1\cdots x_n y^6z^6y^2z^2 \overset{(4.2)}{\approx} x_1\cdots x_n y^7z^7yz \\
&\overset{(4.1)}{\approx} x_1\cdots x_n yzyz \overset{(4.3)}{\approx} x_1\cdots x_n y^2z^2, \\
(\mathbf{v}\varphi_3)y^2z^2 &\overset{(4.1)}{\approx} x_1\cdots x_n z^6y^2z^2 \overset{(4.2)}{\approx} x_1\cdots x_n z^7y^2z \\
&\overset{(4.1)}{\approx} x_1\cdots x_n zy^2z.
\end{aligned}
$$

Hence $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\} = \mathbf{S}_3\{\hat{\mathbf{u}}\approx\mathbf{v}, (9.2_n^2), (9.2_\infty^2), (9.2_\dagger^2)\}$ by (9.11). Since

$$
\begin{aligned}
\hat{\mathbf{u}} &\overset{(4.1)}{\approx} x_1\cdots x_n y^7z^7\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3'' \\
&\overset{(4.3)}{\approx} \big(x_1\cdots x_n y^6z^2\big)yz^5\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3'' \\
&\overset{(9.2_n^2)}{\approx} x_1\cdots x_n\big(zy^6zyz^5\mathbf{u}_2\mathbf{u}_3'z\big)\mathbf{u}_3'' \\
&\overset{(4.2)}{\approx} x_1\cdots x_n z^7y^7\mathbf{u}_2\mathbf{u}_3'z\mathbf{u}_3'' \\
&\overset{(4.1)}{\approx} \mathbf{u}^\star,
\end{aligned}
$$

it follows that $\mathbf{S}_3\{\mathbf{u}\approx\mathbf{v}\} = \mathbf{S}_3\{\mathbf{u}^\star\approx\mathbf{v}, (9.2_n^2), (9.2_\infty^2), (9.2_\dagger^2)\}$. $\square$

Proof of Lemma D. This is achieved by applying Lemmas 9.4, 9.5, and 9.6, and their dual results. Let $\mathbf{u}\approx\mathbf{v}$ be any balanced identity that is not a permutation identity. If the simple letters of $\mathbf{u}$ do not appear in the same order as the simple letters of $\mathbf{v}$, then by Lemma 9.4, they can be put in the same order modulo some identities from $Ⓢ\cup\{(9.1), (9.2), (9.3)\}$, resulting in an identity $\mathbf{u}^{(1)}\approx\mathbf{v}^{(1)}$ that satisfies property (P) in Lemma 9.5.

If $y\prec_{\mathbf{u}^{(1)}} z$ and $z\prec_{\mathbf{v}^{(1)}} y$ for some simple letter y and some non-simple letter z, then by Lemma 9.5, the letter z in $\mathbf{u}^{(1)}$ can be moved to the left until it immediately precedes y, modulo some identities from $Ⓢ\cup\{(9.2)\}$. This procedure can be repeated to obtain an identity $\mathbf{u}^{(2)}\approx\mathbf{v}^{(2)}$ that satisfies property (Q) in Lemma 9.6.

If $y\prec_{\mathbf{u}^{(2)}} z$ and $z\prec_{\mathbf{v}^{(2)}} y$ for some non-simple letters y and z, then by Lemma 9.6, the letter z in $\mathbf{u}^{(2)}$ can be moved to the left until it immediately precedes y, modulo some identities from $Ⓢ\cup\{(9.2)\}$. This procedure can be repeated to obtain an identity $\mathbf{u}^{(3)}\approx\mathbf{v}^{(3)}$ that satisfies the property $p\prec_{\mathbf{u}} q$ if and only if $p\prec_{\mathbf{v}} q$ for any letters p and q. Hence the identity $\mathbf{u}^{(3)}\approx\mathbf{v}^{(3)}$ is ip-compliant.

Note that whenever any of Lemmas 9.5 and 9.6 is invoked, only the first occurrence of some non-simple letter from a word is moved to the left; this does not alter the multiplicity of any letter and the final part of the word. Therefore the identity $\mathbf{u}^{(3)}\approx\mathbf{v}^{(3)}$ is balanced with $\mathsf{fp}(\mathbf{u}^{(3)}) = \mathsf{fp}(\mathbf{u})$ and $\mathsf{fp}(\mathbf{v}^{(3)}) = \mathsf{fp}(\mathbf{v})$.

Now the identity $\mathbf{u}^{(3)}\approx\mathbf{v}^{(3)}$ satisfies property (P) in Lemma 9.5. By arguments that are dual to those in the second and third paragraphs of this proof, the dual results of Lemmas 9.5 and 9.6 can be applied to $\mathbf{u}^{(3)}\approx\mathbf{v}^{(3)}$ to obtain an identity $\mathbf{u}'\approx\mathbf{v}'$ that is fp-compliant with $\mathsf{ip}(\mathbf{u}') = \mathsf{ip}(\mathbf{u}^{(3)})$ and $\mathsf{ip}(\mathbf{v}') = \mathsf{ip}(\mathbf{v}^{(3)})$. Since $\mathsf{ip}(\mathbf{u}^{(3)}) = \mathsf{ip}(\mathbf{v}^{(3)})$, the identity $\mathbf{u}'\approx\mathbf{v}'$ is also ip-compliant. $\square$

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